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DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS

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RADIATION PRESSURE IN THE CONVECTIVE STELLAR MODEL

LOUIS R. HENRICH

ABSTRACT

The investigation of the effect of radiation pressure in the point-source model requires a study of the effect of radiation pressure in the convective model. The equations of state and of equilibrium are, therefore, derived for the wholly convective model with radiation pressure. Tables obtained by integrating these equations for various values of the ratio of the radiation pressure to the gas pressure at the center are given. A relation is also obtained between the mass, the mean molecular weight, and the central value of the ratio of the radiation pressure to the gas pressure.

Since in the limiting case of vanishing radiation pressure the model reduces to the convective polytrope, it is possible to consider the actual density distribution with small amounts of radiation pressure as a perturbation of the polytropic case. The necessary first-order equation is obtained and integrated.

1. *Introduction.*—The effect of radiation pressure in the equation of hydrostatic equilibrium of a star is generally neglected in current discussions of stellar structure. This provides a fair approximation for stars of about the solar mass, as it can be shown¹ quite generally that for these stars the ratio $(1 - \beta_c)$ of the radiation pressure to the total pressure at the center must necessarily be less than about 5 per cent. On the other hand, in masses somewhat larger than that of the sun the radiation pressure will begin to contribute appreciably to the total pressure. In a general way we may expect this to require quite significant corrections to those models in which the radiation pressure is neglected. Thus, it is known that in the luminosity formula derived, for instance, on the standard model² the ratio β of the gas pressure to the total pressure occurs with a fairly high power.

Now, according to our present ideas on the mechanism of energy generation in stars, the energy generation is expected to be so highly concentrated toward the center that a good approximation is provided by the point-source model. All stars, therefore, must be expected to have convective cores. When radiation pressure is neglected, the convective cores are polytropic regions of index $n = 3/2$. However, if radiation pressure cannot be neglected, we can no longer use a polytropic representation of the core. Thus, a first step in the analysis of stellar models with allowance for radiation pressure is the study of wholly convective stars. In this paper, therefore, we shall restrict ourselves to the study of these convective configurations. In a later paper we shall consider the problem of fitting envelopes to such convective cores and thus establish a theoretical relation among the mass, the luminosity, the radius, and the mean molecular weight which is valid for all masses.

2. *The equation of state.*—The equation of convective equilibrium between the total pressure, P , and the density, ρ , can be expressed in the form³

$$\frac{dP}{P} = \Gamma_1 \frac{d\rho}{\rho}, \quad (1)$$

where the adiabatic exponent, Γ_1 , is given by

$$\Gamma_1 = \beta + \frac{(4 - 3\beta)^2(\gamma - 1)}{\beta + 12(\gamma - 1)(1 - \beta)}. \quad (2)$$

¹ S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, p. 75, Chicago, 1939.

² *Ibid.*, p. 233.

³ *Ibid.*, p. 56.

In equation (2) β is the ratio of the gas pressure to the total pressure and γ is the ratio of specific heats, neglecting radiation pressure. Since the light elements, like hydrogen and helium, are highly abundant in stellar interiors, we can assume that $\gamma = 5/3$. Then

$$\Gamma_1 = \frac{3^2 - 24\beta - 3\beta^2}{3(8 - 7\beta)}. \quad (3)$$

On the other hand, we have the general relation

$$P = \left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta}{\beta^4} \right]^{1/3} \rho^{4/3}, \quad (4)$$

where k is the Boltzmann constant, μ is the mean molecular weight which is assumed to be constant, H is the mass of the proton, and a is the Stefan-Boltzmann constant. Differentiating equation (4) logarithmically, we have

$$\frac{dP}{P} = -\frac{4 - 3\beta}{3(1 - \beta)} \frac{d\beta}{\beta} + \frac{4}{3} \frac{d\rho}{\rho}. \quad (5)$$

Eliminating $d\rho/\rho$ between equations (1) and (5) and using equation (3) for Γ_1 we find

$$\frac{dP}{P} \left\{ 1 - \frac{4}{3} \frac{3(8 - 7\beta)}{3^2 - 24\beta - 3\beta^2} \right\} = -\frac{4 - 3\beta}{3(1 - \beta)} \frac{d\beta}{\beta}, \quad (6)$$

or

$$\frac{dP}{P} = -\frac{3^2 - 24\beta - 3\beta^2}{3\beta^2(1 - \beta)} d\beta. \quad (7)$$

The foregoing equation can be integrated to give

$$P = \text{constant } e^{32/3\beta} \beta^{-8/3} (1 - \beta)^{5/3}. \quad (8)$$

Let y denote the ratio of the radiation to the gas pressure, so that

$$y = \frac{1 - \beta}{\beta}; \quad \beta = \frac{1}{1 + y}; \quad 1 - \beta = \frac{y}{1 + y}. \quad (9)$$

In terms of y , equation (8) becomes

$$P = \text{constant } e^{32y/3} y^{5/3} (1 + y). \quad (10)$$

According to equation (4) we have

$$\rho = \text{constant} \left(\frac{\beta^4}{1 - \beta} \right)^{1/4} P^{3/4} = \text{constant} \frac{1}{y^{1/4}(1 + y)^{3/4}} P^{3/4}, \quad (11)$$

or, substituting for P from equation (10), we have

$$\rho = \text{constant } e^{8y} y. \quad (12)$$

Since the gas pressure is given by

$$\beta P = \frac{k}{\mu H} \rho T, \quad (13)$$

we can express T in the form

$$T = \frac{\mu H}{k} \frac{1}{1+y} \frac{P}{\rho}. \quad (14)$$

Then, according to equations (10) and (12), we have

$$T = \text{constant } e^{8y/3} y^{2/3}. \quad (15)$$

We have thus expressed P , ρ , and T in terms of the single parameter y . In terms of the values of these quantities at the center of the configuration we have

$$\left. \begin{aligned} P &= P_c e^{32(y-y_c)/3} \left(\frac{y}{y_c} \right)^{5/3} \left(\frac{1+y}{1+y_c} \right), \\ \rho &= \rho_c e^{8(y-y_c)} \left(\frac{y}{y_c} \right), \\ T &= T_c e^{8(y-y_c)/3} \left(\frac{y}{y_c} \right)^{2/3}, \end{aligned} \right\} \quad (16)$$

which are our fundamental equations.

In addition, the values of the constants in equations (16) may be fixed in terms of any two of the four constants. Thus, let us choose ρ_c and T_c to be the values of the density and temperature at the center of the configuration. Then the value of y_c will be determined by the equation

$$y_c = \frac{a}{3} \frac{\mu H}{k} \frac{T_c^3}{\rho_c}. \quad (17)$$

This equation follows from the definition of y .

Using equation (13), the first equation of (16) can be written alternatively as

$$P = \frac{k}{\mu H} \rho_c T_c e^{32(y-y_c)/3} \left(\frac{y}{y_c} \right)^{5/3} (1+y). \quad (18)$$

If we let

$$P = P_c e^{-32y_c/3} y_c^{-5/3} (1+y_c)^{-1} \Pi; \quad T = T_c e^{-8y_c/3} y_c^{-2/3} \Theta, \quad (19)$$

where

$$\Pi = e^{32y/3} y^{5/3} (1+y); \quad \Theta = e^{8y/3} y^{2/3}, \quad (20)$$

we obtain expressions for the variation of pressure and temperature which do not involve y_c .

3. *The equation of hydrostatic equilibrium.*—For a spherically symmetric distribution of matter the usual form for the equation is

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho, \quad (21)$$

where r is the radius and G is the gravitational constant. This may be re-written in the present case in terms of the variables y and r . Using the expression for ρ as given by equations (16) and for P as given by equation (18), we have

$$\left. \begin{aligned} \frac{1}{\rho} \frac{dP}{dr} &= \frac{k}{\mu H} T_c e^{-8y_c/3} y_c^{-2/3} e^{-8y} y^{-1} \frac{d}{dr} \{ e^{32y/3} y^{5/3} (1 + y) \}, \\ &= \frac{k}{\mu H} T_c \frac{e^{-8y_c/3}}{y_c^{2/3}} \frac{e^{-8y}}{y} \{ e^{32y/3} [\frac{8}{3} y^{5/3} (1 + y) + \frac{5}{3} y^{2/3} (1 + y) + y^{5/3}] \} \frac{dy}{dr}, \\ &= \frac{k}{\mu H} T_c \frac{e^{-8y_c/3}}{y_c^{2/3}} \{ e^{8y/3} [\frac{8}{3} y^{5/3} + \frac{4}{3} y^{2/3} + \frac{5}{3} y^{-1/3}] \} \frac{dy}{dr}, \\ &= \frac{k}{\mu H} T_c \frac{e^{-8y_c/3}}{y_c^{2/3}} \{ e^{8y/3} [4(\frac{8}{3} y^{5/3} + \frac{5}{3} y^{2/3}) + \frac{5}{2} (\frac{8}{3} y^{2/3} + \frac{2}{3} y^{-1/3})] \} \frac{dy}{dr}, \\ &= \frac{k}{\mu H} T_c \frac{e^{-8y_c/3}}{y_c^{2/3}} \frac{d}{dr} \{ e^{8y/3} y^{2/3} (4y + \frac{5}{2}) \}. \end{aligned} \right\} \quad (22)$$

If we now let

$$F = e^{8(y-y_c)/3} \left(\frac{y}{y_c} \right)^{2/3} (1 + \frac{8}{5}y) \quad (23)$$

and

$$\sigma = e^{8(y-y_c)} \left(\frac{y}{y_c} \right), \quad (24)$$

then equation (21) becomes⁴

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) = -\frac{8\pi G}{5} \frac{\mu H}{k} \frac{\rho_c}{T_c} \sigma. \quad (25)$$

Now introduce the variable, ξ , defined by

$$r = a\xi; \quad a^2 = \frac{5k}{8\pi G \mu H} \frac{T_c}{\rho_c}; \quad (26)$$

then equation (25) becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{dF}{d\xi} \right) = -\sigma. \quad (27)$$

⁴ This equation is essentially the same as that given by J. Wasiutynski in *M.N.*, 100, 364, 1940. (There appears to be a typographical error in his equation 4.) Since the present writer was not aware of any place in which the complete equations were derived, it seemed useful to set down the derivations.

On examining equations (23) and (24), it appears that for very small values of the quantity y_c the only significant terms in the expressions for F and σ are $(y/y_c)^{2/3}$ and (y/y_c) , respectively. Thus, as y_c approaches zero, equation (27) approaches the Lane-Emden equation of index $n = 3/2$. As the quantity y_c becomes very large, the dominant terms in the expressions for F and σ are $e^{8(y-y_c)/3}$ and $e^{8(y-y_c)}$, respectively. So for increasing values of y_c , equation (27) approaches the Lane-Emden equation of index $n = 3$.

We now have both F and σ expressed parametrically in terms of the quantities y and y_c . For purposes of actually integrating equation (27) it would be convenient to eliminate y from the relation between F and σ . This elimination cannot be carried out analytically. However, the elimination can be effected numerically by tabulating σ as a function of F . Since both F and σ involve y_c , this would strictly mean a tabulation with two parameters in the form $\sigma(F, y_c)$. However, y_c occurs in this relation in a simple way so that we can eliminate it from the tabulation. To do this we shall introduce Z and ξ instead of F and σ , defined as follows:

$$F = e^{-8y_c/3} y_c^{-2/3} Z; \quad \sigma = e^{-8y_c} y_c^{-1} \xi, \quad (28)$$

where

$$Z = e^{8y/3} y^{2/3} (1 + \frac{8}{3}y); \quad \xi = e^{8y} y. \quad (29)$$

Further, let

$$\xi = a_1 \eta; \quad a_1 = e^{8y_c/3} y_c^{1/6}. \quad (30)$$

In terms of these variables equation (27) becomes

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{dZ}{d\eta} \right) = -\xi, \quad (31)$$

in which form the explicit occurrence of y_c has been eliminated.

The following relation between the derivatives in the two different sets of variables may be noted:

$$\frac{dF}{d\xi} = \frac{e^{-8y_c/3} y_c^{-2/3}}{e^{8y_c/3} y_c^{1/6}} \frac{dZ}{d\eta} = \frac{a_1}{\xi_c} \frac{dZ}{d\eta}, \quad (32)$$

where ξ_c is the value of ξ at the center of the configuration.

4. *The mass relation and other relations.*—We have the following equation for the mass

$$M(y_c) = \int_0^R 4\pi \rho r^2 dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \sigma \xi^2 d\xi, \quad (33)$$

or

$$M(y_c) = 4\pi a^3 \rho_c \left(-\xi^2 \frac{dF}{d\xi} \right)_{\xi_1}, \quad (34)$$

where R is the total radius of the star and the bracketed quantity is evaluated at the boundary, where ξ takes the value ξ_1 . In terms of the Z, η variables this relation is

$$M(y_c) = \frac{4\pi a^3 a_1^3 \rho_c}{\xi_c} \left(-\eta^2 \frac{dZ}{d\eta} \right)_{\eta_1}. \quad (35)$$

According to equations (26) and (17) the quantity $a^3 \rho_c$ in equation (34) is given by

$$a^3 \rho_c = \left(\frac{5k}{8\pi G \mu H} \right)^{3/2} \left(\frac{T_c}{\rho_c} \right)^{3/2} \rho_c = \left(\frac{5k}{8\pi G \mu H} \right)^{3/2} \left(\frac{3k}{a \mu H} y_c \right)^{1/2}. \quad (36)$$

The mass relation (34) then becomes

$$M(y_c) = C \left(\frac{5}{2} \right)^{3/2} \frac{1}{\mu^2} y_c^{1/2} \left(-\xi^2 \frac{dF}{d\xi} \right)_{\xi_1}, \quad (37)$$

where we have put

$$C = 4\pi \left(\frac{1}{4\pi G} \right)^{3/2} \left(\frac{3}{a} \right)^{1/2} \left(\frac{k}{H} \right)^2. \quad (38)$$

Inserting numerical values for the various physical constants we find $M(y_c)$ in solar masses to be

$$\frac{M(y_c)}{\odot} = 4.420 \frac{1}{\mu^2} y_c^{1/2} \left(-\xi^2 \frac{dF}{d\xi} \right)_{\xi_1}. \quad (39)$$

We may also obtain limiting expressions for the mass by considering the two limiting polytropes $n = 3/2$ and $n = 3$. We have⁵

$$M(n) = 4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \left(-\xi^2 \frac{d\theta_n}{d\xi} \right)_{\xi_1}, \quad (40)$$

where θ_n is the solution of the Lane-Emden equation with index n , and K is defined by the relation

$$P = K \rho^{1+1/n}. \quad (41)$$

Using equation (4) we may express K in terms of β_c and ρ_c as follows:

$$K = \left[\left(\frac{k}{\mu H} \right)^4 \frac{3}{a} \frac{1 - \beta_c}{\beta_c^4} \right]^{1/3} \rho_c^{(n-3)/3n}. \quad (42)$$

Then, substituting in equation (40), we have

$$M(n) = C(n+1)^{3/2} \frac{1}{\mu^2} \left(\frac{1 - \beta_c}{\beta_c^4} \right)^{1/2} \left(-\xi^2 \frac{d\theta_n}{d\xi} \right)_{\xi_1}, \quad (43)$$

or in terms of y_c we have

$$M(n) = C(n+1)^{3/2} \frac{1}{\mu^2} y_c^{1/2} (1 + y_c)^{3/2} \left(-\xi^2 \frac{d\theta_n}{d\xi} \right)_{\xi_1}. \quad (44)$$

With $n = 3/2$ and 3, respectively, the foregoing equation gives the limiting forms of equation (39) for the cases $y_c = 0$ and $y_c = \infty$. Then

$$\frac{M(n = \frac{3}{2})}{\odot} = 12.00 \frac{1}{\mu^2} y_c^{1/2} (1 + y_c)^{3/2}, \quad (45)$$

⁵ Chandrasekhar, *op. cit.*, p. 97.

and

$$\frac{M(n=3)}{\odot} = 18.05 \frac{1}{\mu^2} y_c^{1/2} (1 + y_c)^{3/2}. \quad (46)$$

The ratio of the central to the mean density is easily seen to be

$$\frac{\rho_c}{\bar{\rho}} = \frac{1}{3} \left(\frac{-\xi}{\frac{dF}{d\xi}} \right)_{\xi_1}. \quad (47)$$

From equation (26) we can further obtain a relation between the quantity y_c , the central temperature, and the radius of the configuration. We have

$$T_c = \frac{8\pi G \mu H}{5k} \frac{a^3 \rho_c \xi_1}{a \xi_1}. \quad (48)$$

Using equation (36), we find

$$T_c = \left(\frac{5}{8\pi G} \frac{3}{a} \right)^{1/2} \frac{k}{\mu H} \frac{y_c^{1/2} \xi_1}{R}. \quad (49)$$

When this is combined with equation (37), we have

$$T_c = Q(y_c) \frac{1}{1 + y_c} \frac{\mu H}{k} \frac{GM(y_c)}{R}, \quad (50)$$

where

$$Q(y_c) = \frac{2}{5} \frac{1 + y_c}{\left(-\xi \frac{dF}{d\xi} \right)_{\xi_1}}. \quad (51)$$

For the two limiting polytropes there is a similar relation in which

$$Q(n) = \frac{1}{(n + 1) \left(-\xi \frac{d\theta_n}{d\xi} \right)_{\xi_1}}. \quad (52)$$

We also see that the quantities

$$\left(\frac{5}{2} \frac{1}{1 + y_c} \right)^{1/2} \xi_1; \quad \left(\frac{5}{2} \frac{1}{1 + y_c} \right)^{1/2} \left(-\frac{dF}{d\xi} \right)_{\xi_1} \quad (53)$$

lie, respectively, between the values of the quantities

$$(n + 1)^{1/2} \xi_1(n); \quad (n + 1)^{1/2} \left(-\frac{d\theta_n}{d\xi} \right)_{\xi_1} \quad (54)$$

for the two limiting polytropes.

5. *Integration of the equations and results.*—We have seen in equations (16) how the physical structure of the wholly convective configurations are uniquely determined in terms of y_c . A one-parametric series of integrations of the final equation will therefore suffice to determine the properties of these configurations. The integration was started in each case by using a series expansion for Z and ζ , as follows:

$$Z = Z_c + Z_2\eta^2 + Z_4\eta^4 + \dots, \quad (55)$$

and

$$\zeta(Z) = \zeta(Z_c) + (Z - Z_c)\left(\frac{d\zeta}{dZ}\right)_{Z_c} + \frac{(Z - Z_c)^2}{2!}\left(\frac{d^2\zeta}{dZ^2}\right)_{Z_c} + \dots \quad (56)$$

Substituting these series in the differential equation (31) and comparing coefficients, we find

$$\left. \begin{aligned} Z = Z_c - \frac{1}{6}\zeta_c\eta^2 + \frac{1}{120}(\zeta_c\zeta'_c)\eta^4 - \frac{1}{7!}(\zeta_c\zeta_c'^2 + \frac{5}{8}\zeta_c^2\zeta_c'')\eta^6 \\ + \frac{1}{9!}(\zeta_c\zeta_c'^3 + \frac{3}{8}\zeta_c^2\zeta_c'\zeta_c'' + \frac{3}{9}\zeta_c^3\zeta_c''')\eta^8 + \dots, \end{aligned} \right\} \quad (57)$$

where the primes denote differentiation with respect to Z . If $\zeta = Z^n$, it is seen that this relation reduces to the ordinary starting series for the Lane-Emden equation of index n .

TABLE 1
CONSTANTS OF THE CONFIGURATIONS

Model	$\frac{\xi_1}{(1+y_c)^{1/2}}$	$\left(-\frac{dF}{d\xi}\right)_{\xi_1} \frac{1}{(1+y_c)^{1/2}}$	$\frac{M}{\odot} \mu^2$	$\frac{\rho_c}{\bar{\rho}}$	Q
$n = 3/2 \dots\dots$	3.654*	0.2033†		5.99	0.538
$y_c = 0.01 \dots\dots$	3.681	.2019	1.23	6.08	.538
$y_c = 0.10 \dots\dots$	3.914	.1889	4.67	6.91	.541
$y_c = 0.25 \dots\dots$	4.260	.1693	9.49	8.39	.555
$y_c = 0.5 \dots\dots$	4.738	.1449	18.68	10.90	.582
$y_c = 1.0 \dots\dots$	5.446	.1165	43.2	15.58	.630
$y_c = 2.0 \dots\dots$	6.311	.0918	118.7	22.93	.691
$n = 3 \dots\dots$	8.724*	0.0537†		54.18	0.854

* = $[\frac{2}{3}(n+1)]^{1/2}\xi_1(n)$.

† = $[\frac{2}{3}(n+1)]^{1/2}(-d\theta_n/d\xi)_{\xi_1}$.

The values of Z , Π , Θ , ζ , and α_1 are given in Table 9 as functions of y . The value of ζ as a function of Z is given in Table 10. This latter table was used in performing all the integrations except the one for $y_c = 2.0$. The results of the integrations are given in Tables 3–8, where η , Z , $dZ/d\eta$, and ζ are tabulated for each step of the integration. In the case of $y_c = 2.0$, Table 10 was not used since the values of Z and ζ became too large. A separate table was developed for this case, the integration giving ξ , F , $dF/d\xi$, σ , and y . Integrations have been performed for $y_c = 0.01, 0.10, 0.25, 0.5, 1.0, 2.0$.

The more important quantities characterizing the configurations are collected in Table 1. The values for the limiting polytropes $n = 3/2$ and 3 are also given.

TABLE 2

PERTURBING FUNCTION

ξ	f_1	σ_1	ξ	f_1	σ_1
0.0	1.60000	0.00000	1.9	1.54837	0.38095
0.1	1.60000	.00000	2.0	1.53388	.41845
0.2	1.60000	.00013	2.1	1.51678	.45253
0.3	1.60000	.00066	2.2	1.49692	.48222
0.4	1.59999	.00203	2.3	1.47417	.50667
0.5	1.59997	.00484	2.4	1.44847	.52519
0.6	1.59991	.00971	2.5	1.41979	.53723
0.7	1.59979	.01731	2.6	1.38815	.54240
0.8	1.59953	.02825	2.7	1.35364	.54046
0.9	1.59908	.04305	2.8	1.31639	.53129
1.0	1.59833	.06207	2.9	1.27658	.51486
1.1	1.59714	.08547	3.0	1.23446	.49118
1.2	1.59536	.11320	3.1	1.19031	.46019
1.3	1.59280	.14500	3.2	1.14447	.42169
1.4	1.58925	.18036	3.3	1.09732	.37500
1.5	1.58449	.21858	3.4	1.04932	.31848
1.6	1.57827	.25880	3.5	1.00099	.24766
1.7	1.57034	.30000	3.6	0.95296	0.14587
1.8	1.56045	0.34109			

TABLE 3

 $\eta_1 = 7.7603$ $y_c = 0.01$

η	Z	$-dZ/d\eta$	ζ	η	Z	$-dZ/d\eta$	ζ
0.0	0.048433	0.0000000	0.0108329	4.0	0.026038	0.0085341	0.0042294
0.2	.048361	.0007212	.0108083	4.2	.024334	.0085069	.0038185
0.4	.048145	.0014365	.0107349	4.4	.022638	.0084421	.0034244
0.6	.047787	.0021402	.0106136	4.6	.020959	.0083427	.0030488
0.8	.047290	.0028266	.0104460	4.8	.019303	.0082118	.0026933
1.0	.046658	.0034903	.0102342	5.0	.017676	.0080524	.0023588
1.2	.045896	.0041264	.0099809	5.2	.016084	.0078680	.0020464
1.4	.045009	.0047303	.0096892	5.4	.014530	.0076616	.0017564
1.6	.044006	.0052978	.0093626	5.6	.013020	.0074366	.0014892
1.8	.042893	.0058253	.0090051	5.8	.011557	.0071960	.0012449
2.0	.041679	.0063099	.0086207	6.0	.010143	.0069430	.0010232
2.2	.040372	.0067489	.0082137	6.2	.008780	.0066805	.0008238
2.4	.038982	.0071407	.0077884	6.4	.007471	.0064115	.0006464
2.6	.037519	.0074837	.0073494	6.6	.006216	.0061385	.0004905
2.8	.035992	.0077776	.0069008	6.8	.005016	.0058644	.0003554
3.0	.034411	.0080219	.0064469	7.0	.003870	.0055916	.0002409
3.2	.032787	.0082174	.0059917	7.2	.002779	.0053226	.0001466
3.4	.031128	.0083647	.0055390	7.4	.001741	.0050596	.0000726
3.6	.029444	.0084654	.0050923	7.6	.000754	.0048056	.0000209
3.8	0.027745	0.0085211	0.0046548	η_1	0.000000	0.0046101	0.0000000

TABLE 4

 $\eta_1 = 4.6153$ $y_c = 0.1$

η	Z	$-dZ/d\eta$	ζ	η	Z	$-dZ/d\eta$	ζ
0.0	0.326290	0.000000	0.222554	2.4	0.167696	0.096512	0.074742
.1	.325919	.007410	.222129	2.5	.158078	.095795	.068006
.2	.324809	.014769	.220861	2.6	.148546	.094795	.061592
.3	.322968	.022027	.218763	2.7	.139128	.093537	.055510
.4	.320409	.029136	.215860	2.8	.129847	.092048	.049770
.5	.317148	.036048	.212188	2.9	.120725	.090350	.044375
.6	.313207	.042721	.207789	3.0	.111783	.088468	.039327
.7	.308613	.049113	.202713	3.1	.103037	.086427	.034625
.8	.303395	.055187	.197015	3.2	.094502	.084249	.030262
0.9	.297587	.060912	.190759	3.3	.086191	.081956	.026235
1.0	.291225	.066259	.184011	3.4	.078114	.079569	.022533
1.1	.284349	.071204	.176838	3.5	.070280	.077108	.019147
1.2	.276999	.075729	.169312	3.6	.062694	.074592	.016068
1.3	.269218	.079819	.161503	3.7	.055362	.072039	.013284
1.4	.261050	.083467	.153481	3.8	.048287	.069466	.010783
1.5	.252539	.086667	.145315	3.9	.041469	.066889	.008555
1.6	.243731	.089419	.137070	4.0	.034909	.064322	.006589
1.7	.234670	.091728	.128808	4.1	.028604	.061779	.004874
1.8	.225400	.093601	.120587	4.2	.022552	.059274	.003405
1.9	.215964	.095050	.112460	4.3	.016748	.056819	.002175
2.0	.206404	.096088	.104476	4.4	.011186	.054427	.001185
2.1	.196760	.096733	.096677	4.5	.005860	.052113	.000449
2.2	.187070	.097005	.089101	4.6	.000760	.049892	.000021
2.3	0.177370	0.096924	0.081780	η_1	0.000000	0.049561	0.000000

TABLE 5

 $\eta_1 = 3.0806$ $y_c = 0.25$

η	Z	$-dZ/d\eta$	ζ	η	Z	$-dZ/d\eta$	ζ
0.0	1.08214	0.00000	1.84726	1.6	0.52842	0.47811	0.50836
.1	1.07907	.06138	1.83755	1.7	.48110	.46768	.43178
.2	1.06990	.12161	1.80876	1.8	.43497	.45460	.36279
.3	1.05482	.17956	1.76187	1.9	.39026	.43938	.30131
.4	1.03410	.23424	1.69845	2.0	.34715	.42251	.24704
.5	1.00811	.28474	1.62056	2.1	.30579	.40443	.19963
.6	0.97732	.33034	1.53063	2.2	.26629	.38553	.15861
.7	.94223	.37049	1.43129	2.3	.22870	.36616	.12349
.8	.90341	.40480	1.32528	2.4	.19306	.34664	.09376
0.9	.86147	.43311	1.21527	2.5	.15937	.32722	.06890
1.0	.81699	.45539	1.10378	2.6	.12761	.30813	.04842
1.1	.77059	.47178	0.99309	2.7	.09773	.28956	.03189
1.2	.72282	.48256	.88517	2.8	.06967	.27166	.01889
1.3	.67425	.48810	.78163	2.9	.04337	.25457	.00916
1.4	.62536	.48886	.68375	3.0	.01873	.23842	.00257
1.5	0.57662	0.48535	0.59246	η_1	0.00000	0.22617	0.00000

TABLE 6

$\eta_1 = 1.7171$

$y_c = 0.5$

η	Z	$-dZ/d\eta$	ξ	η	Z	$-dZ/d\eta$	ξ
0.00	4.30175	0.0000	27.2991	0.90	1.92080	3.3538	5.4792
.05	4.29039	0.4535	27.1519	0.95	1.75569	3.2484	4.6067
.10	4.25655	0.8983	26.7157	1.00	1.59615	3.1317	3.8379
.15	4.20085	1.3259	26.0064	1.05	1.44265	3.0068	3.1663
.20	4.12437	1.7287	25.0491	1.10	1.29556	2.8765	2.5847
.25	4.02851	2.1001	23.8765	1.15	1.15506	2.7430	2.0852
.30	3.91498	2.4347	22.5271	1.20	1.02127	2.6086	1.6598
.35	3.78572	2.7286	21.0421	1.25	0.89419	2.4748	1.3007
.40	3.64284	2.9791	19.4641	1.30	.77376	2.3431	1.0005
.45	3.48855	3.1853	17.8339	1.35	.65983	2.2147	0.7521
.50	3.32505	3.3471	16.1899	1.40	.55222	2.0905	.5491
.55	3.15455	3.4660	14.5658	1.45	.45070	1.9712	.3857
.60	2.97913	3.5442	12.9902	1.50	.35502	1.8573	.2566
.65	2.80076	3.5847	11.4863	1.55	.26487	1.7493	.1572
.70	2.62123	3.5911	10.0715	1.60	.17999	1.6474	.0837
.75	2.44215	3.5672	8.7580	1.65	.10003	1.5518	.0331
.80	2.26495	3.5170	7.5533	1.70	.02470	1.4627	.0039
0.85	2.09082	3.4446	6.4604	η_1	0.00000	1.4338	0.0000

TABLE 7

$\eta_1 = 0.53516$

$y_c = 1.0$

η	Z	$-dZ/d\eta$	ξ	η	Z	$-dZ/d\eta$	ξ
0.00	37.4190	0.000	2980.96	0.28	14.9985	90.363	385.86
.01	37.3693	9.919	2971.95	.29	14.1080	87.725	337.50
.02	37.2210	19.729	2945.13	.30	13.2442	85.022	294.05
.03	36.9755	29.328	2901.05	.31	12.4077	82.278	255.16
.04	36.6355	38.615	2840.65	.32	11.5987	79.516	220.47
.05	36.2045	47.497	2765.14	.33	10.8174	76.754	189.65
.06	35.6871	55.894	2676.04	.34	10.0636	74.010	162.35
.07	35.0885	63.734	2575.04	.35	9.3371	71.298	138.26
.08	34.4145	70.958	2464.00	.36	8.6374	68.631	117.08
.09	33.6715	77.522	2344.86	.37	7.9643	66.017	98.53
.10	32.8664	83.393	2219.58	.38	7.3169	63.467	82.35
.11	32.0061	88.552	2090.08	.39	6.6947	60.986	68.30
.12	31.0977	92.993	1958.20	.40	6.0969	58.579	56.14
.13	30.1486	96.722	1825.65	.41	5.5228	56.252	45.69
.14	29.1656	99.753	1693.99	.42	4.9716	54.006	36.75
.15	28.1558	102.110	1564.59	.43	4.4424	51.844	29.16
.16	27.1255	103.827	1438.63	.44	3.9345	49.767	22.76
.17	26.0812	104.940	1317.11	.45	3.4468	47.775	17.41
.18	25.0286	105.492	1200.80	.46	2.9787	45.868	12.99
.19	23.9731	105.529	1090.32	.47	2.5292	44.046	9.39
.20	22.9196	105.098	986.08	.48	2.0975	42.307	6.50
.21	21.8726	104.247	888.38	.49	1.6828	40.650	4.25
.22	20.8359	103.024	797.33	.50	1.2842	39.073	2.54
.23	19.8132	101.477	712.97	.51	0.9011	37.574	1.32
.24	18.8073	99.650	635.20	.52	0.5325	36.152	0.52
.25	17.8209	97.587	563.86	0.53	0.1778	34.803	0.08
.26	16.8562	95.327	498.71	η_1	0.0000	34.135	0.00
0.27	15.9149	92.907	439.49				

TABLE 8

$\xi_1 = 10.9308$

$\gamma_c = 2.0$

ξ	F	$-dF/d\xi$	σ	γ
0.00	4.20000	0.00000	1.00000	2.00000
0.25	4.18960	.08302	0.99379	1.99927
0.50	4.15864	.16420	.97543	1.99707
0.75	4.10780	.24178	.94574	1.99344
1.00	4.03818	.31421	.90598	1.98839
1.25	3.95124	.38018	.85778	1.98196
1.50	3.84872	.43868	.80299	1.97420
1.75	3.73257	.48905	.74356	1.96516
2.00	3.60490	.53094	.68142	1.95490
2.25	3.46781	.56432	.61833	1.94349
2.50	3.32343	.58942	.55587	1.93099
2.75	3.17376	.60667	.49534	1.91746
3.00	3.02070	.61671	.43777	1.90296
3.25	2.86595	.62024	.38390	1.88756
3.50	2.71105	.61808	.33422	1.87132
3.75	2.55731	.61103	.28898	1.85428
4.00	2.40587	.59989	.24826	1.83650
4.25	2.25765	.58543	.21197	1.81801
4.50	2.11338	.56835	.17992	1.79885
4.75	1.97363	.54929	.15187	1.77904
5.00	1.83885	.52882	.12748	1.75860
5.25	1.70930	.50742	.10643	1.73755
5.50	1.58518	.48551	.08837	1.71587
5.75	1.46656	.46343	.07296	1.69357
6.00	1.35346	.44148	.05990	1.67061
6.25	1.24579	.41987	.04888	1.64697
6.50	1.14348	.39879	.03963	1.62261
6.75	1.04634	.37837	.03190	1.59744
7.00	0.95423	.35871	.02548	1.57141
7.25	.86692	.33988	.02018	1.54440
7.50	.78421	.32193	.01582	1.51629
7.75	.70588	.30487	.01226	1.48690
8.00	.63170	.28872	.00938	1.45603
8.25	.56144	.27347	.00706	1.42341
8.50	.49489	.25910	.00522	1.38868
8.75	.43182	.24559	.00377	1.35138
9.00	.37203	.23291	.00264	1.31086
9.25	.31530	.22102	.00179	1.26623
9.50	.26146	.20990	.00115	1.21616
9.75	.21029	.19949	.00069	1.15855
10.00	.16166	.18977	.00038	1.08990
10.25	.11536	.18069	.00017	1.00342
10.50	.07126	.17221	.00006	0.88338
10.75	.02920	.16430	.00001	0.67310
ξ_1	0.00000	0.15891	0.00000	0.00000

TABLE 9

y	Z	Π	Θ	ζ	α_1
0.02	0.080204	0.0018605	0.077717	0.023470	0.549542
.04	.138454	.0074547	.130126	.055085	.650631
.06	.197121	.0184859	.179854	.096964	.734253
.08	.259230	.0376564	.229814	.151718	.812515
.10	.326290	.0688610	.281284	.222554	.889498
.12	.399366	.1176026	.335039	.313404	0.967174
.14	.479369	.191572	.391641	.429080	1.046706
.16	.567155	.301431	.451557	.575462	1.128892
.18	.663578	.461865	.515201	.759725	1.21434
.20	.769516	0.692989	.582967	0.990606	1.30355
.22	0.885886	1.022214	.655241	1.27874	1.39698
.24	1.013659	1.48674	.732413	1.63703	1.49503
.26	1.15387	2.13683	.814879	2.08116	1.59811
.28	1.30762	3.04018	.903050	2.63013	1.70660
.30	1.47609	4.28767	0.997355	3.30695	1.82091
.32	1.66054	6.00089	1.098242	4.13946	1.94144
.34	1.86235	8.34214	1.206182	5.16131	2.06859
.36	2.08296	11.52747	1.321674	6.41314	2.20279
.38	2.32395	15.8438	1.44524	7.94399	2.34449
.40	2.58701	21.6713	1.57744	9.81301	2.49416
.42	2.87395	29.5128	1.71887	12.09146	2.65227
.44	3.18672	40.0319	1.87014	14.8651	2.81934
.46	3.52741	54.1029	2.03192	18.2373	2.99590
.48	3.89828	72.8761	2.20491	22.3322	3.18252
.50	4.30175	97.8615	2.38986	27.2991	3.37977
.52	4.74041	131.040	2.58756	33.3172	3.58830
.54	5.21706	175.003	2.79885	40.6019	3.80875
.56	5.73469	233.144	3.02463	49.4114	4.04183
.58	6.29653	309.889	3.26583	60.0557	4.28825
.60	6.90602	411.014	3.52348	72.9063	4.54880
.62	7.56689	544.046	3.79864	88.4082	4.82428
.64	8.28310	718.777	4.09244	107.0946	5.11555
.66	9.05894	947.941	4.40610	129.6041	5.42353
.68	9.89898	1248.07	4.74089	156.7007	5.74917
.70	10.80814	1640.63	5.09818	189.298	6.09349
.72	11.7917	2153.43	5.47941	228.491	6.45755
.74	12.8553	2822.50	5.88612	275.585	6.84247
.76	14.0050	3694.45	6.31993	332.142	7.24946
.78	15.2473	4829.56	6.78259	400.030	7.67977
.80	16.5891	6305.73	7.27592	481.476	8.13473
.82	18.0380	8223.50	7.80189	579.143	8.61575
.84	19.6018	10712.61	8.36256	696.207	9.12430
.86	21.2893	13940.3	8.96013	836.459	9.66196
.88	23.1094	18122.0	9.59694	1004.421	10.23037
.90	25.0722	23535.2	10.27547	1205.49	10.83130
.92	27.1879	30536.8	10.99835	1446.09	11.4666
.94	29.4680	39585.9	11.7684	1733.89	12.1382
.96	31.9245	51272.4	12.5885	2078.04	12.8481
0.98	34.5702	66354.0	13.4619	2489.40	13.5986
1.00	37.4190	85803.4	14.3919	2980.96	14.3919
0.01	0.048433	0.0005216	0.047670	0.010833	0.476703
0.25	1.082142	1.78482	0.772959	1.84726	1.54592
2.00	$1.38093 \cdot 10^3$	$8.76510 \cdot 10^9$	$3.28794 \cdot 10^2$	$8.88611 \cdot 10^6$	$2.32492 \cdot 10^2$

TABLE 10

Z	y	ξ	Z	y	ξ
0.001	0.000031616	0.000031624	0.05	0.01045865	0.01137137
.002	.000089392	.000089456	.06	.01348626	.01502269
.003	.000164144	.000164359	.07	.01665630	.01903042
.004	.000252574	.000253084	.08	.01993232	.023337813
.005	.000352756	.000353753	.09	.02328539	.02805339
.006	.000463382	.000465103	.10	.02669235	.03304650
.007	.000583479	.000586209	.11	.03013452	.03834973
.008	.000712288	.000716358	.12	.03359686	.04395681
.009	.000849188	.000854977	.13	.03706719	.04986252
.010	.000993663	.001001593	.14	.04053567	.05606253
.011	.001145267	.001155809	.15	.04399438	.06255316
.012	.001303617	.001317283	.16	.04743691	.06933125
.013	.001468370	.001485721	.17	.05085814	.07639408
.014	.001639223	.001660861	.18	.05425399	.08373930
.015	.001815902	.001842474	.19	.05762121	.09136482
.016	.001998156	.002030353	.20	.06095730	.09926887
.017	.002185758	.002224314	.21	.06426027	.1074498
.018	.002378496	.002424188	.22	.06752866	.1159062
.019	.002576176	.002629820	.23	.07076137	.1246368
.020	.002778615	.002841072	.24	.07395764	.1336405
.021	.002985642	.003057813	.25	.07711696	.1429163
.022	.003197100	.003279927	.26	.08023908	.1524632
.023	.003412837	.003507300	.27	.08332388	.1622805
.024	.003632710	.003739832	.28	.08637141	.1723673
.025	.003856587	.003977427	.29	.08938184	.1827232
.026	.004084336	.004219995	.30	.09235544	.1933473
.027	.004315845	.004467459	.31	.09529256	.2042393
.028	.004550979	.004719724	.32	.09819362	.2153986
.029	.004789645	.004976732	.33	.1010591	.2268248
.030	.005031729	.005238408	.34	.1038895	.2385176
.031	.005277131	.005504685	.35	.1066853	.2504764
.032	.005525753	.005775505	.36	.1094471	.2627012
.033	.005777501	.006050804	.37	.1121754	.2751915
.034	.006032285	.006330532	.38	.1148709	.2879471
.035	.006290017	.006614630	.39	.1175342	.3009678
.036	.006550613	.006903051	.40	.1201659	.3142536
.037	.006813992	.007195746	.41	.1227665	.3278038
.038	.007080075	.007492670	.42	.1253366	.3416188
.039	.007348788	.007793778	.43	.1278769	.3556982
0.040	0.007620057	0.008099030	0.44	0.1303879	0.3700420

TABLE 10 - Continued

Z	y	ξ	Z	y	ξ
0.45	0.1328702	0.3846499	4	0.4852027	23.53369
.46	.1353244	.3995222	5	.5310969	37.18720
.47	.1377510	.4146585	6	.5696482	54.29591
.48	.1401506	.4300590	7	.6029444	75.01028
.49	.1425238	.4457236	8	.6322850	99.47089
.50	.1448710	.4616521	9	.6585353	127.8101
.6	.1670260	.6354650	10	.6823018	160.1533
.7	.1870903	.8357373	11	.7040266	196.6202
.8	.2054231	1.062582	12	.7240421	237.3251
0.9	.2223031	1.316151	13	.7426047	282.3779
1.0	.2379487	1.596621	14	.7599168	331.8847
1.1	.2525332	1.904184	15	.7761405	385.9478
1.2	.2661962	2.239041	16	.7914082	444.6662
1.3	.2790512	2.601400	17	.8058292	508.1366
1.4	.2911922	2.991471	18	.8194949	576.4522
1.5	.3026973	3.409468	19	.8324825	649.7042
1.6	.3136325	3.855601	20	.8448579	727.9817
1.7	.3240541	4.330086	21	.8566775	811.3713
1.8	.3340100	4.833134	22	.8679903	899.9579
1.9	.3435418	5.364956	23	.8788393	993.8249
2.0	.3526858	5.925760	24	.8892618	1093.053
2.1	.3614737	6.515756	25	.8992908	1197.723
2.2	.3699332	7.135148	26	.9089558	1307.913
2.3	.3780892	7.784140	27	.9182829	1423.699
2.4	.3859636	8.462937	28	.9272955	1545.158
2.5	.3935759	9.171732	29	.9360146	1672.363
2.6	.4009439	9.910728	30	.9444592	1805.388
2.7	.4080835	10.68012	31	.9526464	1944.305
2.8	.4150088	11.48010	32	.9605918	2089.184
2.9	.4217351	12.31086	33	.9683098	2240.098
3.0	.4282681	13.17260	34	.9758130	2397.112
3.1	.4346246	14.06549	35	.9831136	2560.297
3.2	.4408126	14.98972	36	.9902224	2729.718
3.3	.4468410	15.94549	37	.9971495	2905.443
3.4	.4527184	16.93296	38	1.003904	3087.538
3.5	0.4584523	17.95232	39	1.010495	3276.063
			40	1.016930	3471.089

Figure 1 illustrates the density distribution for the various cases integrated and also the density distribution for the two limiting polytropes. In Figure 2 the logarithm of the product of the mass, in units of the solar mass, times the square of the mean molecular weight is plotted against the logarithm of y_c . It is seen that this curve lies between the two limiting curves given by equations (45) and (46).

6. *Perturbation theory for small values of y_c .*—It has been indicated that in the limiting case of vanishing radiation pressure the model considered here reduces simply to the polytrope with index $n = 3/2$. Consequently, it would seem likely that the density

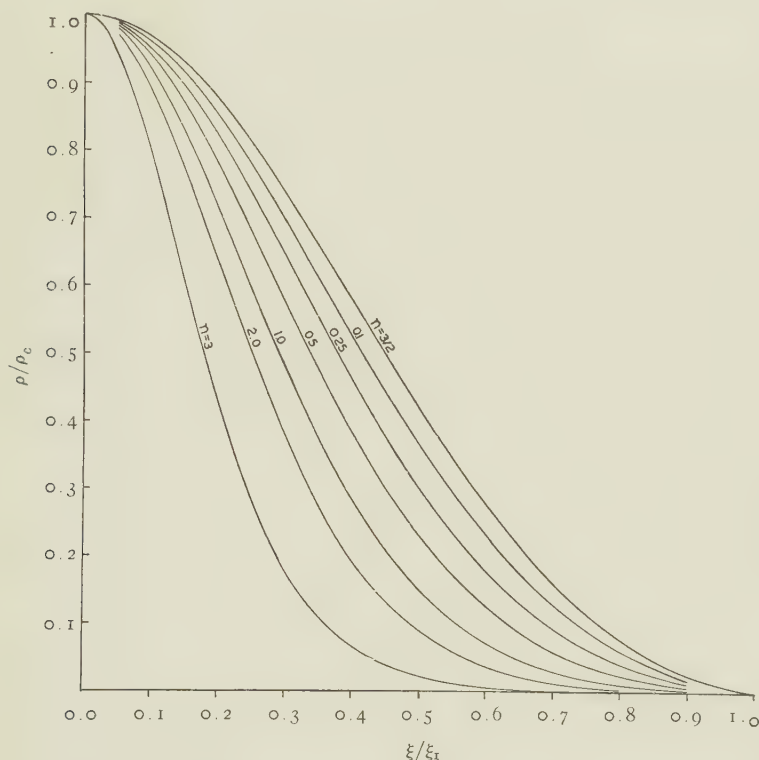


FIG. 1.—The density distribution in the convective model with radiation pressure. Each curve is labeled with the central value of the ratio of the radiation pressure to the gas pressure (y_c). The two limiting curves are the solutions of the Lane-Emden equation with index $n = \frac{3}{2}$ and $n = 3$, respectively.

distribution in stars of small mass might be represented quite accurately by considering the extent to which the density distribution in these stars deviated from the distribution in the limiting polytrope. It would, therefore, be of interest to consider the effect of radiation pressure as a perturbation of the polytrope of index $n = 3/2$. Such a calculation would probably be sufficient for the treatment of the majority of stars.

We shall, therefore, seek a solution of the differential equation of the following form

$$F = \theta_{3/2} + y_c f_1; \quad (58)$$

and

$$\sigma = \theta_{3/2}^{3/2} + y_c \sigma_1, \quad (59)$$

where $\theta_{3/2}$ denotes the Lane-Emden function of index $n = 3/2$, while f_1 and σ_1 are first-order terms for the potential and density, respectively. We shall restrict ourselves to

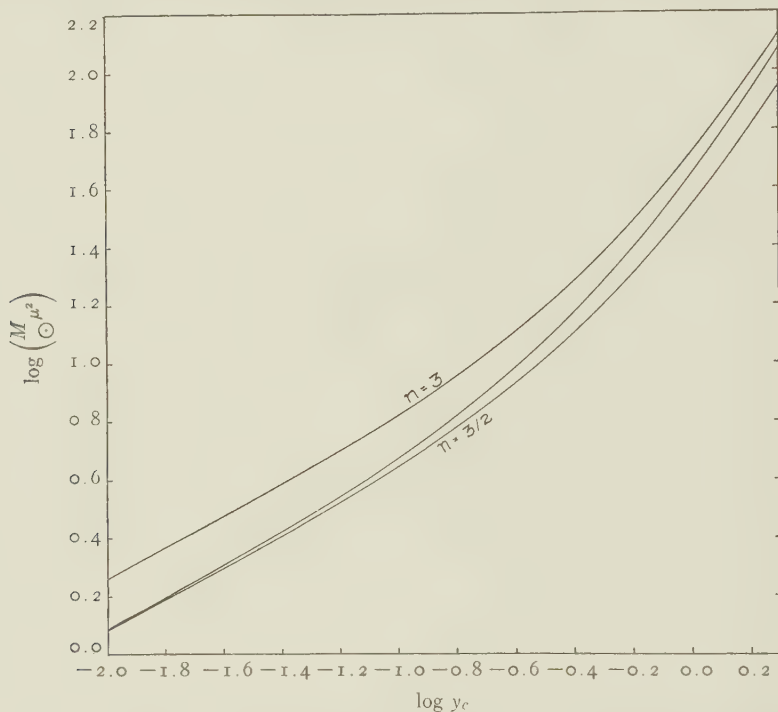


FIG. 2.—The mass as a function of y_c . The central curve is for the convective model with radiation pressure. The asymptotes are for the polytropes of index $n = \frac{3}{2}$ and $n = 3$, respectively.

terms of the first order and neglect the higher-order terms. Using equations (23) and (24) we find

$$\left. \begin{aligned} F &= \left(\frac{y}{y_c} \right)^{2/3} \left(1 + \frac{64}{15} y - \frac{8}{3} y_c + \dots \right), \\ \sigma &= \left(\frac{y}{y_c} \right) (1 + 8[y - y_c] + \dots) \end{aligned} \right\} \quad (60)$$

or, eliminating y between the foregoing equations, we find

$$\sigma = F^{3/2} + y_c \left(\frac{8}{3} F^3 - 4 F^{3/2} \right) + \dots \quad (61)$$

Expanding F in terms of $\theta_{3/2}$ and f_1 , according to equation (58), and retaining only the first-order terms, we have

$$\sigma = \theta_{3/2}^{3/2} + y_c \left(\frac{8}{3} \theta_{3/2}^{1/2} f_1 - 4 \theta_{3/2}^{3/2} + \frac{2}{5} \theta_{3/2}^3 \right), \quad (62)$$

so that

$$\sigma_1 = \frac{3}{2}\theta_{3/2}^{1/2}f_1 - 4\theta_{3/2}^{3/2} + \frac{8}{5}\theta_{3/2}^3. \quad (63)$$

On substituting equations (58) and (59) in the differential equation (27) and eliminating the terms which represent merely the solution of the Lane-Emden equation, we obtain

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{df_1}{d\xi} \right) = -\frac{3}{2}\theta_{3/2}^{1/2}f_1 + 4\theta_{3/2}^{3/2} - \frac{8}{5}\theta_{3/2}^3. \quad (64)$$

Actually, this relation is not valid near the boundary. For it was necessary to expand the quantity $F^{3/2}$ as a binomial series in which $\theta_{3/2}$ was taken as the larger term. Since $\theta_{3/2}$ approaches zero at the boundary, this condition is not fulfilled for all values of the radius. But for sufficiently small values of y_c the errors are not appreciable except in the immediate neighborhood of the boundary.

The results of the integration of equation (64) are given in Table 2. Using this, a comparison was made with the exact integration for $y_c = 0.01$. This showed that up to 97 per cent of the radius this method gave the quantity σ with an error of less than two units in the fourth decimal place. On the other hand, a comparison with the integration for $y_c = 0.1$ indicated that this perturbation theory would probably not be useful for values of $y_c \geq 0.03$.

We may finally note here that this perturbation theory can be used in connection with the complete point-source model to obtain the first-order corrections introduced by radiation pressure in the mass-luminosity relation obtained, for instance, on the basis of the Cowling model.

I am greatly indebted to Dr. S. Chandrasekhar for suggesting this investigation and for many helpful discussions.

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RADIATION PRESSURE IN THE POINT-SOURCE STELLAR MODEL

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ABSTRACT

The effect of radiation pressure in a point-source stellar model is investigated. Envelopes in radiative equilibrium with the opacity following a corrected Kramers law are fitted to cores in convective equilibrium for four different values of the ratio of radiation pressure to gas pressure at the center. The theoretical mass-luminosity-radius relation is obtained and is applied to determine the hydrogen content of a few stars. The model appears to be a possible one for stars of less than ten times the solar mass. It does not appear to be applicable to the Trumpler stars.

1. *Introduction.*—The purpose of the present study is to investigate the effect of radiation pressure in a stellar model having a core in convective equilibrium. If y_c represents the ratio of radiation pressure to gas pressure at the center of the model, a separate integration of the core and envelope equations is needed for each value of y_c . The equations for the convective cores have already been integrated and published¹ by the present author for six different values of y_c . In the present paper four of the models are completed by fitting radiative envelopes to the convective cores.

2. *The law of opacity.*—One of the first problems in integrating the envelope equations is the question of the exact form for the law of opacity in the equation of radiative equilibrium. The law assumed is Kramers' formula corrected by the Gaunt and guillotine factors, which is then written as

$$\kappa = \kappa_0 \rho T^{-3.5} \left(\frac{\bar{g}}{\tau} \right), \quad (1)$$

where κ_0 is a constant depending on the chemical composition, ρ the density, T the temperature, $\bar{g}$ the mean Gaunt factor, and τ the guillotine factor. Since $(\bar{g}/\tau)$ cannot be expressed in a simple manner in terms of ρ and T , it is necessary to use an approximate expression for it.

To determine this approximate expression the following procedure was adopted. A star of the solar mass and radius was assumed to be built on the Cowling² model. On the basis of this assumption the actual temperature and density distribution in the model may be computed. Using these values of T and ρ and assuming an appropriate chemical composition, we may obtain from the tables of P. M. Morse³ the values of $\log(\tau/\bar{g})$. On plotting these against $\log \theta$, where $\theta = T/T_c$ and T_c is the central temperature, it is found that for a certain part of the envelope region between the core and a certain value of θ , say θ_κ , we may assume that

$$\frac{\tau}{\bar{g}} = 10 \theta, \quad (2)$$

and for values of $\theta < \theta_\kappa$ we may assume that

$$\frac{\tau}{\bar{g}} = 10 \theta_\kappa = \text{const.} \quad (3)$$

¹ *Ap. J.*, 93, 483, 1941. This paper will be referred to as "paper I" and the equations therein as "I:n."

² Convective core and radiative envelope with no radiation pressure. Integration given in *M.N.*, 96, 42, 1936.

³ *Ap. J.*, 92, 27, 1940.

Thus, the law of opacity may be taken as

$$\left. \begin{aligned} \kappa &= \frac{\kappa_0 \rho T^{-3.5}}{10 \theta_\kappa} & (\theta < \theta_\kappa) \\ &= \frac{\kappa_0 \rho T^{-3.5}}{10 \theta} & (\theta \geq \theta_\kappa). \end{aligned} \right\} \quad (4)$$

After the integration for each value of y_c is completed, the general correctness of the expression used for $(\tau/\bar{g})$ and the value chosen for θ_κ may be tested. This is done by taking a star of the appropriate mass and using its radius to fix the actual temperature and density in the model. Then, with these values of T and ρ we again enter Morse's tables to determine $\log (\tau/\bar{g})$.

3. *The envelope equations.*—The equilibrium equations are

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2} \rho; \quad (5) \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho, \quad (6)$$

and

$$\frac{d}{dr} \left(\frac{1}{3} a T^4 \right) = -\frac{L}{4\pi c r^2} \kappa_0 \rho^2 T^{-3.5} \left(\frac{\bar{g}}{\tau} \right), \quad (7)$$

where the total pressure

$$P = \frac{k}{\mu H} \rho T + \frac{1}{3} a T^4. \quad (8)$$

Here L is the luminosity and is assumed constant in the envelope. The rest of the symbols have the same meaning as in paper I.

Let us make the following substitutions:

$$\rho = \rho_c \sigma = \rho_c a_2^{-3} s; \quad (9) \quad T = T_c \theta = T_c a_2^{-1} t, \quad (10)$$

$$r = a \xi = a a_2 l; \quad (11) \quad M(r) = M_0 \psi = 4\pi a^3 \rho_c \psi, \quad (12)$$

where a is given by

$$a^2 = \frac{5k}{8\pi G \mu H} \frac{T_c}{\rho_c} \quad (13)$$

and where ρ_c is the central density. Then we have

$$P = \frac{k}{\mu H} a_2^{-4} \rho_c T_c \bar{\varphi}^4, \quad (14)$$

where

$$\bar{\varphi}^4 = st + y_c t^4; \quad \varphi^4 = \sigma \theta + y_c \theta^4, \quad (15)$$

and

$$y_c = \frac{a}{3} \frac{\mu H}{k} \frac{T_c^3}{\rho_c}. \quad (16)$$

Transforming equations (5) and (6), we find

$$\frac{d\bar{\varphi}^4}{dl} = -\frac{5}{2} \frac{\psi s}{l^2}; \quad (17) \quad \frac{d\psi}{dl} = l^2 s. \quad (18)$$

Equation (7) becomes

$$\frac{dt}{dl} = -\frac{3\kappa_0 L}{16\pi a c} \frac{\rho_c^2 a_2^{0.5}}{\alpha T_c^{7.5}} \frac{10\theta_\kappa s^2 \bar{g}}{l^2 l^{6.5} \tau} \quad (19)$$

or

$$\frac{dt}{dl} = -Q_0 \frac{10\theta_\kappa s^2 \bar{g}}{l^2 l^{6.5} \tau}, \quad (20)$$

where

$$Q_0 = \frac{3\kappa_0 L \rho_c^2 a_2^{0.5}}{160\pi a c \alpha T_c^{7.5} \theta_\kappa}. \quad (21)$$

With the law of opacity assumed, we shall have for the inner part of the envelope

$$\frac{dt}{dl} = -Q_0 \frac{\theta_\kappa s^2}{l^2 l^{6.5} \theta} = -Q_0 t_\kappa \frac{s^2}{l^2 l^{7.5}} = -Q_1 \frac{s^2}{l^2 l^{7.5}}, \quad (22)$$

where

$$Q_1 = Q_0 t_\kappa \quad (23)$$

and where t_κ is the value of t at the point where the law of opacity changes. Hence, we shall have

$$\left. \begin{aligned} \frac{dt}{dl} &= -Q_0 \frac{s^2}{l^2 l^{6.5}} & (t < t_\kappa) \\ &= -Q_1 \frac{s^2}{l^2 l^{7.5}} & (t \geq t_\kappa) \end{aligned} \right\} \quad (24)$$

4. *The equations of fit.*—The next problem to be considered is how the envelope is to be fitted to the core. The usual conditions⁴ for any sort of a fitting are that at the interface the mass, pressure, temperature, and radius should be the same for both the core and the envelope functions. In addition, in the present case there is the condition that at the interface the radiative gradient becomes unstable and the convective region begins. At this point, therefore, the effective polytropic indices for core and envelope must equal, so that we have

$$\left(\frac{d \log T}{d \log P} \right)_{\text{env}} = \left(\frac{d \log T}{d \log P} \right)_{\text{core}}, \quad (25)$$

where the subscripts "env" and "core" denote that the quantity in parenthesis is evaluated on the envelope and on the core side of the interface, respectively. If we assume that the mean molecular weight is constant throughout the star, it will also be true that ρ is continuous across the interface. Hence, it follows from equation (5) that dP/dr will be equal on both sides of the interface, and we may write equation (25) as

$$\left(\frac{P}{T} \frac{dT}{dr} \frac{dr}{dP} \right)_{\text{env}} = \left(\frac{P}{T} \frac{dT}{dr} \frac{dr}{dP} \right)_{\text{core}} \quad (26)$$

or

$$\left(\frac{dT}{dr} \right)_{\text{env}} = \left(\frac{dT}{dr} \right)_{\text{core}}. \quad (27)$$

Instead of using the relation as it stands for the condition of the equality of the effective polytropic indices, it is more convenient to use it to determine the quantity Q_1 , for the

⁴ See, e.g., S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, p. 171, Chicago: University of Chicago Press, 1939.

envelope, in terms of certain of the variables on the core side of the interface. We have from equation (24)

$$Q_1 = -\frac{l^2 t^{7.5}}{s^2} \frac{dt}{dl} = -a_2^{1.5} \frac{\xi^2 \theta^{7.5}}{\sigma^2} \frac{d\theta}{d\xi}; \quad (28)$$

or, letting

$$Q_2 = Q_1 a_2^{-1.5}, \quad (29)$$

we have

$$Q_2 = -\frac{\xi^2 \theta^{7.5}}{\sigma^2} \frac{d\theta}{d\xi}. \quad (30)$$

In addition to this condition, we may derive two others which will be independent of scale factors. They are

$$U_{\text{env}} = U_{\text{core}}; \quad (31a) \quad V_{\text{env}} = V_{\text{core}}, \quad (31b)$$

where

$$U = \frac{4\pi\rho r^3}{M(r)}; \quad V = \frac{2}{5} \frac{GM(r)\rho}{r p_g}, \quad (32)$$

and where p_g is the gas pressure. Introducing equations (9), (11), and (12) into equation (32), we have

$$U_{\text{env}} = \left(\frac{s l^3}{\psi} \right)_{\text{env}}; \quad V_{\text{env}} = \left(\frac{\psi}{l t} \right)_{\text{env}}, \quad (33)$$

and

$$U_{\text{core}} = \left(\frac{\sigma \xi^3}{\psi} \right)_{\text{core}}; \quad V_{\text{core}} = \left(\frac{\psi}{\xi \theta} \right)_{\text{core}}. \quad (34)$$

Hence, our four equations of fit are (30), (31a), (31b), and the equality of the masses

$$\psi_{\text{env}} = \psi_{\text{core}}. \quad (35)$$

The core functions which are derived from the integrations of paper I have been expressed, for the purpose of fitting, in terms of the variables ξ , σ , θ , ψ , and Q_2 .

5. *The starting series for integrating inward from the boundary.*—It is readily seen that in starting the integrations from the outer boundary we must choose ψ_R , Q_2 , and a_2 so as to have a suitable fit at the interface. (The subscript R indicates that the quantity is to be evaluated at the boundary of the radiative envelope.) Hence, the solution of the equations of equilibrium in this manner requires essentially a three-parameter set of integrations to give a physically possible model. If the law of opacity is retained in the usual Kramers form, in which it is not necessary to change it for a certain value of θ , it will require only a two-parameter family of integrations. In this case it is not necessary to choose a_2 .

If the integration is made from the center outward, the integration for each value of y_c can be reduced to making a one-parameter set of integrations—by starting integrating at varying distances from the center of the core function. However, the condition for the outer boundary of the model is that ρ and T should tend to zero simultaneously. When one attempts to integrate the equations with such boundary conditions, it is found that it is very difficult to determine the boundary accurately.

Consequently, in spite of the apparent increase in the amount of computation involved in determining an "eigenvalue" from a three-parameter set of integrations instead of from a one-parameter set of integrations, it was decided to integrate from the outer boundary inward. The method for starting the integration follows and is based on a procedure indicated in the monograph by Chandrasekhar.⁵ However, the method

⁵ *Ibid.*, p. 295.

has been extended so as to give at least five-place accuracy in the starting values for integration.

In developing the necessary starting series it is assumed that the mass, or ψ , remains constant. This is, of course, perfectly proper, as we can always restrict the region in which the starting series is applied so that ψ does not change in the last place retained.

Now let us re-write equation (24) as

$$\frac{dt^4}{dl} = -4Q_0 \frac{s^2}{l^2 t^{3.5}}. \quad (36)$$

Then, dividing equation (17) by equation (36), we have

$$\frac{d(st + y_c t^4)}{dt^4} = \frac{5}{8Q_0} \frac{\psi_R t^{3.5}}{s} \quad (37)$$

or

$$y_c + \frac{d(st)}{dt^4} = \frac{5\psi_R}{8Q_0} \frac{t^{3.5}}{s} \quad (38)$$

or

$$\frac{1}{y_c} \frac{d(st)}{dt^4} = \frac{5\psi_R}{8Q_0 y_c} \frac{t^{3.5}}{s} - 1. \quad (39)$$

Now let

$$s = \frac{1}{2Q_0^8} \left(\frac{5\psi_R}{4y_c} \right)^{15} s'; \quad t = \frac{1}{Q_0^2} \left(\frac{5\psi_R}{4y_c} \right)^4 t' \quad (40)$$

and

$$K = \frac{1}{Q_0^2 y_c} \left(\frac{5\psi_R}{4y_c} \right)^3. \quad (41)$$

Introducing these substitutions into equations (39) and (36), we have

$$\frac{K}{2} \frac{d(s't')}{dt'^4} = \frac{t'^{3.5}}{s'} - 1. \quad (42)$$

and

$$\frac{dt'^4}{dl} = -\frac{s'^2}{l^2 t'^{3.5}}. \quad (43)$$

Making the further substitutions

$$t' = \frac{x^2}{K^2}; \quad s' = z \frac{x^6}{K^7}, \quad (44)$$

we have

$$\frac{1}{16} z x \frac{dz}{dx} = x - z \left(1 + \frac{z}{2} \right) \quad (45)$$

and

$$\frac{dx}{dl} = -\frac{K}{8} \frac{z^2}{l^2 x^2}. \quad (46)$$

Letting

$$\delta = \frac{1}{16}, \quad (47)$$

equation (45) becomes

$$\delta z x \frac{dz}{dx} = x - z \left(1 + \frac{z}{2} \right). \quad (48)$$

We shall now obtain an expansion for z as a power series in δ . To do this, let

$$z = z_0 + \delta z_1 + \delta^2 z_2 + \delta^3 z_3 + \dots \quad (49)$$

Substituting this in equation (48), we find

$$\left. \begin{aligned} & \delta x z_0 \frac{dz_0}{dx} + \delta^2 x \left(z_1 \frac{dz_0}{dx} + z_0 \frac{dz_1}{dx} \right) + \delta^3 x \left(z_2 \frac{dz_0}{dx} + z_1 \frac{dz_1}{dx} + z_0 \frac{dz_2}{dx} \right) + \dots \\ & = x - z_0 \left(1 + \frac{z_0}{2} \right) - \delta [z_1 (1 + z_0)] - \delta^2 [z_2 (1 + z_0) + \frac{1}{2} z_1^2] \\ & \quad - \delta^3 [z_3 (1 + z_0) + z_2 z_1] + \dots \end{aligned} \right\} \quad (50)$$

Equating coefficients of equal powers of δ , we have

$$x = z_0 \left(1 + \frac{z_0}{2} \right) \quad (51)$$

or

$$z_0 = \sqrt{1 + 2x} - 1 = \omega - 1, \quad (52)$$

where

$$\omega = \sqrt{1 + 2x}. \quad (53)$$

Proceeding in this manner and letting

$$Z_n = \frac{z_n}{\omega - 1}, \quad (54)$$

we find

$$Z = \frac{z}{\omega - 1} = 1 + \frac{1}{2} \delta Z_1 + \frac{1}{8} \delta^2 Z_2 + \frac{1}{16} \delta^3 Z_3 + \frac{1}{128} \delta^4 Z_4 + \dots, \quad (55)$$

where

$$\left. \begin{aligned} Z_1 &= -\frac{\omega^2 - 1}{\omega^2}, \\ Z_2 &= +\frac{\omega^2 - 1}{\omega^2} \left(\frac{\omega - 1}{\omega} \right) \frac{3\omega^2 + 4\omega + 5}{\omega^2}, \\ Z_3 &= -\frac{\omega^2 - 1}{\omega^2} \left(\frac{\omega - 1}{\omega} \right)^2 \frac{5\omega^4 + 13\omega^3 + 25\omega^2 + 39\omega + 30}{\omega^4}, \\ Z_4 &= +\frac{\omega^2 - 1}{\omega^2} \left(\frac{\omega - 1}{\omega} \right)^3 \frac{35\omega^6 + 132\omega^5 + 331\omega^4 + 648\omega^3 + 1281\omega^2 + 1908\omega + 1105}{\omega^6}. \end{aligned} \right\} \quad (56)$$

Substituting for x and z in equation (46), we have

$$\int \frac{\omega(\omega + 1)^2}{Z^2} d\omega = -\frac{K}{2} \int \frac{dl}{l^2} = W(\omega). \quad (57)$$

We must now integrate this expression inward from the outer boundary. At the boundary

$$l = 1; \quad x = 0; \quad \omega = 1. \quad (58)$$

Hence,

$$W(\omega) = -\frac{K}{2} \int_1^l \frac{dl}{l^2} = \frac{K}{2} \left(\frac{1}{l} - 1 \right). \quad (59)$$

Or, if

$$u = \frac{1}{l}, \quad (60)$$

then

$$u = 1 + \frac{2}{K} W(\omega). \quad (61)$$

To integrate the other side of equation (57), we must expand Z^{-2} as a series in δ and integrate term by term. We find that

$$W(\omega) = \int_1^\omega \frac{\omega(\omega+1)^2}{Z^2} d\omega = W_0(\omega) + \delta W_1(\omega) + \frac{1}{4} \delta^2 W_2(\omega) + \dots, \quad (62)$$

where

$$\left. \begin{aligned} W_0(\omega) &= \frac{\omega^4}{4} + \frac{2}{3} \omega^3 + \frac{1}{2} \omega^2 - \frac{1}{2} \omega, \\ W_1(\omega) &= W_0(\omega) + \frac{1}{2} - \left(\frac{1}{2} \omega^2 + 2\omega + \log_e \omega \right), \\ W_2(\omega) &= \frac{3}{8} - \left(\frac{1}{3} \omega^3 + 3\omega^2 + 3\omega - 12 \log_e \omega + \frac{9}{\omega} - \frac{3}{\omega^2} - \frac{5}{3\omega^3} \right). \end{aligned} \right\} \quad (63)$$

Introducing the variables ω and Z into equations (40), we have

$$s = \frac{2^{11}}{10^6} y_c^{13} \frac{Q_0^6}{\psi_R^6} (\omega+1)^6 (\omega-1)^7 Z \quad (64)$$

and

$$t = \frac{2^4}{10^2} y_c^4 \frac{Q_0^2}{\psi_R^2} (\omega+1)^2 (\omega-1)^2. \quad (65)$$

The integration is started by computing both Z and W to the required accuracy as functions of ω . This tabulation can be made once and for all. In starting any particular integration, y_c , Q_0 , and ψ_R are chosen, thus determining K . Then, using equation (61), we have the relation between the radial factor and the parameter ω . The values of s and t at the chosen values of u are then determined by equations (64) and (65).

With u as the variable representing the radial factor, the equations to be integrated become

$$\left. \begin{aligned} \frac{d\bar{\varphi}}{du} &= \frac{5}{8} \frac{\psi s}{\varphi^3}; & \frac{d\psi}{du} &= -\frac{s}{u^4}, \\ \frac{dt}{du} &= \frac{10 Q_0 \theta_\kappa \bar{g}}{\tau} \frac{s^2}{t^{6.5}}. \end{aligned} \right\} \quad (66)$$

Owing to the nature of the differential equations to be solved, it is found more convenient to do the integration in terms of logarithmic variables. Using an asterisk (*) to denote the logarithm to the base 10, we have

$$s^* = \log s; \quad t^* = \log t; \quad \psi^* = \log \psi; \quad u^* = \log u; \quad \bar{\varphi}^* = \log \bar{\varphi}. \quad (67)$$

Then

$$\frac{d\bar{\varphi}^*}{du^*} = \frac{u}{\varphi} \frac{d\bar{\varphi}}{du}, \quad (68)$$

etc., so that the equations to be integrated become

$$\left. \begin{aligned} \frac{d\bar{\varphi}^*}{du^*} &= \frac{5}{8} \frac{u\psi s}{\varphi^4} = 10^{[-0.20412+u^*-4\bar{\varphi}^*+\psi^*+s^*]}, \\ \frac{d\psi^*}{du^*} &= -\frac{s}{\psi u^3} = -10^{[-3u^*-\psi^*+s^*]}, \\ \frac{dt^*}{du^*} &= \frac{10Q_0\theta_\kappa\bar{g}}{\tau} \frac{u s^2}{t^{7.5}} = \frac{10Q_0\theta_\kappa\bar{g}}{\tau} 10^{[u^*-7.5\theta^*+2\sigma^*]}. \end{aligned} \right\} \quad (69)$$

6. *The mass-luminosity-radius and other relations.*—Similar to the mass relation found in the earlier paper (I:39), we now obtain

$$M = 4\pi\alpha^3\rho_c\psi_R = \left(\frac{5}{2}\right)^{3/2} C \frac{y_c^{1/2}}{\mu^2} \psi_R, \quad (70)$$

where

$$C = \left(\frac{1}{G}\right)^{3/2} \left(\frac{3}{4\pi a}\right)^{1/2} \left(\frac{k}{H}\right)^2. \quad (71)$$

With the mass measured in solar units we have

$$\mu^2 M = 4.420 y_c^{1/2} \psi_R. \quad (72)$$

The ratio of the central to the mean density is given by

$$\frac{\rho_c}{\bar{\rho}} = \frac{1}{3} \frac{\alpha_2^3}{\psi_R}; \quad (73)$$

or, measuring the mass and the radius in solar units,

$$\log \rho_c = \log \left(\frac{1}{3} \frac{\alpha_2^3}{\psi_R} \right) + \log \frac{M}{R^3} + 0.149. \quad (74)$$

To determine the central temperature T_c we have from equation (13)

$$T_c = \frac{8\pi G\mu H}{5k} a^2 \rho_c = \frac{2}{5} \frac{\alpha_2}{\psi_R} \frac{\mu H}{k} \frac{GM}{R}; \quad (75)$$

or, measuring the mass and the radius in solar units, we have

$$\log T_c = 6.966 + \log \frac{\alpha_2}{\psi_R} + \log \frac{\mu M}{R}. \quad (76)$$

We may obtain the alternative expression

$$T_c = \left(\frac{5}{2}\right)^{1/2} C \frac{GH}{k} \frac{\alpha_2 y_c^{1/2}}{\mu R} = \left(\frac{15}{8\pi Ga}\right)^{1/2} \frac{k}{H} \frac{\alpha_2 y_c^{1/2}}{\mu R}; \quad (77)$$

and, measuring the radius in solar units,

$$\log T_c = 7.611 + \log (\alpha_2 y_c^{1/2}) - \log (\mu R). \quad (78)$$

To obtain the mass-luminosity-radius relation we start from equation (21) and find

$$L = \frac{160\pi a c}{3\kappa_0} \frac{\alpha T_c^{7.5} Q_0 \theta_\kappa}{\rho_c^2 \alpha_2^{0.5}}. \quad (79)$$

Using equations (23), (29), (16), and (77), this relation becomes, in terms of solar units,

$$\log (L\mu^{3.5}\kappa_0 R^{0.5}) = \log C_3 + \log (y_c^{2.75} \alpha_2^{0.5} Q_2), \quad (80)$$

where

$$C_3 = 160 \pi c \left(\frac{5}{8 \pi G} \right)^{0.75} \left(\frac{3}{a} \right)^{1.75} \left(\frac{k}{H} \right)^{3.5} \frac{1}{L_{\odot} R_{\odot}^{0.5}} = 10^{32.291}. \quad (81)$$

(The luminosity and radius of the sun, $L_{\odot}$ and $R_{\odot}$, are to be evaluated in c.g.s. units, or in the same units in which the physical constants are evaluated.)

We obtain the more usual type of expression for L by substituting for y_c in terms of $\mu^2 M$ and thus find that

$$L = \frac{10 \cdot 4^4 \pi^3 c a}{3} G^{7.5} \left(\frac{H}{k} \right)^{7.5} \left(\frac{2}{5} \right)^{7.5} \frac{a_2^{0.5} Q_2}{\psi_R^{5.5}} \frac{\mu^{7.5} M^{5.5}}{\kappa_0 R^{0.5}}; \quad (82)$$

or, in solar units,

$$\log L = 28.742 + \log \left(\frac{a_2^{0.5} Q_2}{\psi_R^{5.5}} \right) + \log \left(\frac{\mu^{7.5} M^{5.5}}{\kappa_0 R^{0.5}} \right). \quad (83)$$

7. *Integration of the equations; application of results to determine hydrogen content; etc.*—The envelope equations were integrated for values of $y_c = 0.01, 0.1, 0.25$, and 0.5 . In

TABLE 1
CONSTANTS OF THE CONFIGURATIONS

CONSTANTS	y_c			
	0.01	0.1	0.25	0.5
ξ_i	1.083	1.343	1.689	2.180
$\xi_R = a_2$	8.956	10.18	14.14	43.23
$q = \xi_i / \xi_R$	0.121	0.132	0.119	0.050
ψ_i	0.356	0.628	1.125	2.11
ψ_R	3.41	3.96	4.94	6.82
$\nu = \psi_i / \psi_R$	0.104	0.158	0.228	0.309
Q_2	0.150	0.212	0.301	0.426
θ_K	0.158	0.158	0.128	0.101
$\mu^2 M = 4.420 y_c^{1/2} \psi_R \dagger$	1.51	5.54	10.9	21.3
$\rho_c / \rho = a_2^3 / 3 \psi_R$	70.2	88.8	191	3946
$T_c / \frac{\mu H}{k} \frac{GM}{R} = \frac{2}{5} \frac{a_2}{\psi_R} \dagger$	1.050	1.028	1.144	2.534
$\log \left(L / \frac{\mu^{7.5} M^{5.5}}{\kappa_0 R^{0.5}} \right) = 28.742 + \log \left(\frac{a_2^{0.5} Q_2}{\psi_R^{5.5}} \right) \dagger$	25.461	25.283	24.979	24.603
$\log \{ LR^{0.5} \mu^{1.5} [1 + \mu (1 - \frac{1}{4} Y)] [2\mu (1 - \frac{2}{3} Y) - 1] \}$ $= 7.007 + \log (y_c^{2.75} Q_2 a_2^{0.5}) \dagger$	1.158	4.087	5.406	6.627

† In solar units. ‡ Not in solar units.

making the fit, it was required that the fitted quantities should not differ by more than one unit in the third decimal place of the logarithms. The results of the integrations, together with the core functions, are given in Tables 3–6. Various constants of the configurations are given in Table 1. In this table q and ν represent the fraction of the radius

and the fraction of the mass, respectively, contained in the core. The density distributions for these models have been plotted in Figure 1.

As was mentioned previously, the expression used for the guillotine factor has been tested for certain model stars in the following manner. With the value of μ assumed equal to 1, a value of M is specified for each integration. Then for each of these values of M a mean value of R may be determined⁶ from a plot of the known values of M and R . This value of R determines the actual temperature and density distribution in the model which will determine the correct values of $\log(\tau/\bar{g})$. These values have been calculated for each model and are plotted in Figure 2, together with the assumed values

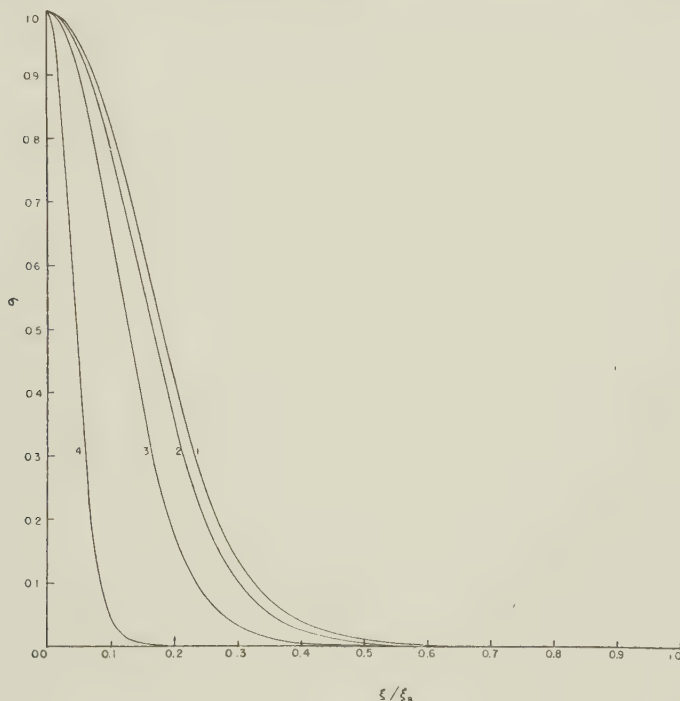


FIG. 1.—Density distribution in the point-source model. The curves 1, 2, 3, and 4 are for the cases $\gamma_c = 0.01, 0.1, 0.25$, and 0.5 , respectively.

that were used in the integrations. It will be seen that the agreement with the assumed expression for the guillotine factor is fairly good for the cases $\gamma_c = 0.01, 0.1$, and 0.25 . In the last case, $\gamma_c = 0.25$, the average error in $\log \kappa$ is probably less than 0.08 , so that this will represent the possible extent of the underestimate of the logarithm of the luminosity. In the case where $\gamma_c = 0.5$ the value of $\log(\tau/\bar{g})$ has probably been underestimated by about 0.6 on the average. However, the main source of opacity in this case must be electron scattering, so that Kramers' formula underestimates the opacity anyway.

The luminosity formula, equation (80) or (83), involves the opacity κ_0 . The evaluation of this quantity is rather complicated, for the absorption per gram of material depends not only on the mixture of heavy elements but also on the proportion of hydrogen to the

⁶ See, e.g., *ibid.*, p. 4.

heavy elements and to helium. Letting X represent the proportion of hydrogen by weight and Y the proportion of helium by weight, $(1 - X - Y)$ will represent the relative abundance of the heavier elements. Then κ_0 will be given by

$$\kappa_0 = 4.32 \times 10^{25} (1 + X) (1 - X - Y), \quad (84)$$

where the numerical constant has been evaluated by B. Strömgren⁷ and also by Morse⁸ for a Russell mixture of heavy elements. If the other quantities in equation (80) are assumed, or known in terms of the hydrogen and helium content, we could consider equation (80) as a means of determining the hydrogen content of a star, for various assumed

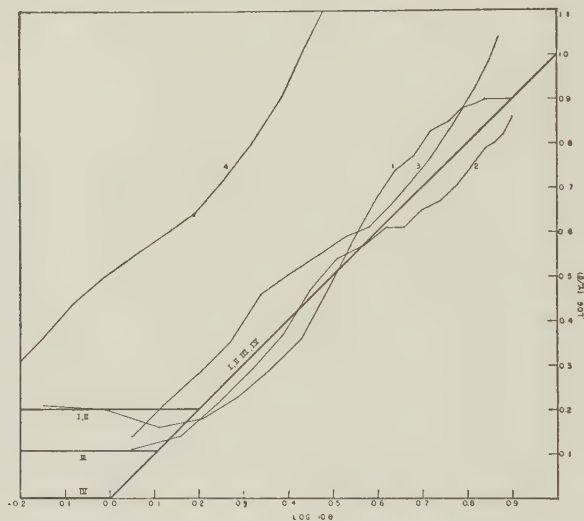


FIG. 2.—The guillotine and Gaunt factors. The curves labeled I, II, III, and IV indicate the assumed expression for $\log(\tau/\xi)$ for the cases $\gamma_e = 0.01, 0.1, 0.25$, and 0.5 , respectively. The curves 1, 2, 3, and 4 show the actual variation of $\log(\tau/\xi)$ in the representative stars for the different values of γ_e .

values of Y . Such calculations have been made before by both Eddington⁹ and B. Strömgren.¹⁰

In case the electron scattering becomes the primary source of opacity, Kramers' law can no longer be used. This point will be considered later.

Let us now introduce equation (84) into equation (80). We find

$$\log(LR^{0.5}) = 6.655 + \log(\gamma_e^{2.75} Q_2 a_2^{0.5}) - \log[\mu^{3.5} (1 + X) (1 - X - Y)]. \quad (85)$$

It is possible to choose μ as the parameter instead of X . We may also assume as a first approximation¹¹ that

$$\mu = \frac{2}{1 + 3X + \frac{1}{2}Y}. \quad (86)$$

⁷ *Zs. f. A p.*, **4**, 118, 1932.

⁸ *Loc. cit.*

⁹ *M.N.*, **92**, 364, 1932.

¹⁰ *Loc. cit.*; *ibid.*, **7**, 222, 1933; *A p. J.*, **87**, 520, 1938.

¹¹ See Chandrasekhar, *op. cit.*, p. 255.

Hence,

$$\log (LR^{0.5}) = 7.007 + \log (y_c^{2.75} Q_2 a_2^{0.5}) - \log \{ \mu^{1.5} [1 + \mu (1 - \frac{1}{4} Y)] [2\mu (1 - \frac{5}{8} Y) - 1] \}. \quad (87)$$

Using this equation and also equation (72), it is possible to plot theoretical relations between $\log (LR^{0.5})$ and $\log M$ valid for the configurations being studied. For each value

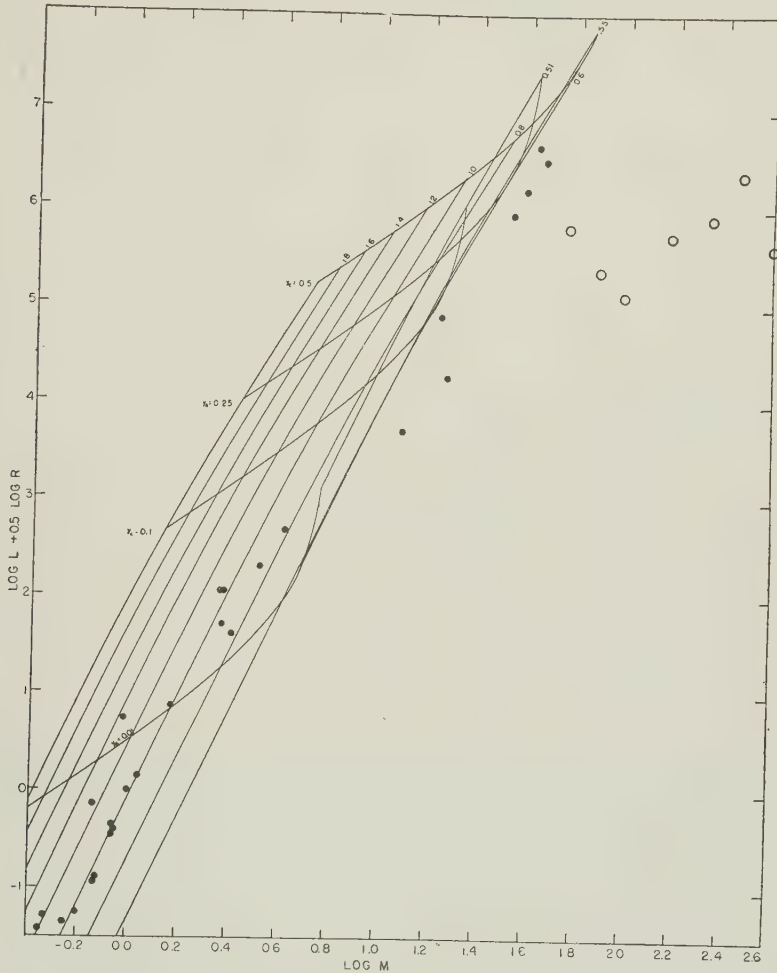


FIG. 3.—The mass-luminosity-radius relation for $\mu = 0.51, 0.55, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8$, and 2.0 with $Y = 0$. The value of μ for the various cases is indicated near the top of the curves. The curves of constant γ_c are also shown. The values of L , M , and R for the stars plotted are taken from Chandrasekhar, *op. cit.*, pp. 489, 490. The open circles are the Trumpler stars.

of Y and μ a curve is obtained in the $(LR^{0.5}, M)$ plane. For $Y = 0$ and for various values of μ between 2 and 0.51, such curves have been plotted in Figure 3. Then, taking the observed values of L , M , and R (in solar units) for various stars, we may represent these stars on the same diagram.

If it is assumed that $Y = 0$ for these stars and that they are built on this model, the value of μ may be read directly from the graph for some of the stars. Then, the value of T_c is directly determined by using equation (78) and interpolating between the solutions for the value of $\log (a_2 y_c^{1/2})$. Similarly, ρ_c is determined by using equation (74).

It will be noticed that, as μ approaches the limiting value ($\mu = 0.50$ for $Y = 0$), the curves for $\mu = \text{const}$ begin to turn back. This is due primarily to the factor $(1 - X - Y)$ in the expression for the opacity. When μ approaches 0.50, X approaches unity. Consequently, the opacity begins to decrease rapidly and the luminosity to increase. This turning-back of these curves may result in the two possible solutions for the hydrogen content that have been mentioned by Eddington and B. Strömgen. However, it is readily seen that the opacity cannot decrease indefinitely in this manner. The electron scattering will provide a lower limit. Consequently, it is evident that two solutions for X are not possible for all stars.

The ratio of the opacity represented by Kramers' formula to the opacity due to electron scattering is given by

$$\frac{\kappa_a}{\kappa_s} \simeq \frac{4.32 \times 10^{25} (1 + X) (1 - X - Y)}{0.19 (1 + X)} \frac{\rho}{T^{3.5}} \frac{\bar{g}}{\tau} \quad (88)$$

or

$$\frac{\kappa_a}{\kappa_s} \simeq 2.3 \times 10^{26} (1 - X - Y) \frac{\rho}{T^{3.5}} \frac{\bar{g}}{\tau} \quad (89)$$

Using the value of T_c and ρ_c as obtained above, we may solve for the ratio (κ_a/κ_s) at some mean value of T , say $T = \frac{2}{3}T_c$, to determine the relative importance of the two sources of opacity. If this ratio is less than about 2 for the larger value of X obtained, it is probable that this second solution for X has no physical significance. Upon making a few calculations it appears that, if stars lie sufficiently close and to the left of the limiting curve, we may expect two possible solutions for X .

The result of setting Y equal to 0.1, 0.2, or 0.4, etc., is to obtain other sets of curves similar to those in Figure 3. As Y takes larger values, the minimum value of μ permissible will also increase. For a fixed value of Y we may again determine μ , X , T_c , and ρ_c for some of the stars. Results for a few stars are given in Table 2. Only the solution for the smaller value of X has been made. In general, the values of X thus determined for any star remain appreciably constant over quite a range in the value of Y . The central temperature also does not vary much, showing a slight decrease in general with increasing Y .

It will be noticed that the massive stars, especially the Trumpler stars, lie to the right of the curves drawn in Figure 3. It is seen that no change in the value of Y will so shift the curves as to include in the region covered by the curves those stars not included in this region for the case $Y = 0$. For, on setting Y equal to any value greater than zero, the limiting curve for small values of μ is moved to the left in the figure. The most obvious conclusion is either that the law of opacity used does not apply to these stars or that the stellar model chosen is not an appropriate one.

As far as the law of opacity is concerned, to include these stars will require increasing the opacity in some manner. Upon investigating the approximate values of T_c and ρ_c it appears that, if electron scattering is taken into account for the stars lying just to the right of the limiting curves, the opacity is generally sufficiently increased to permit a solution with μ somewhere between 0.50 and 0.60. The Trumpler stars, however, lie so far from the permissible regions that it is rather difficult to estimate whether taking into account only the effect of electron scattering will increase the opacity sufficiently to permit considering that they are built on the model considered here. Integration of the envelope equations assuming opacity due only to electron scattering would probably throw some light on the question, as is indicated by Keenan's integrations.¹²

¹² *A. P. J.*, 89, 499, 1939.

8. *Conclusions.*—The integration of the envelope equations has resulted in obtaining the desired theoretical relation between the mass, luminosity, radius, etc. The manner in which the relation changes with y_0 is indicated in Table 1. For stars of less than ten times the solar mass the set of models appears satisfactory, in so far as the theoretical mass-

TABLE 2
SOLUTIONS FOR HYDROGEN CONTENT, CENTRAL
TEMPERATURE, AND DENSITY

STAR		Y			
		0	0.1	0.2	0.4
Sun	μ	1.03	1.00	0.96	0.88
	X	0.31	0.32	0.33	0.36
	$\log T_c$	7.40	7.38	7.26	7.21
	$\log \rho_c$	2.00	1.99	1.99	1.99
Sirius	μ	0.94	0.93	0.89	0.78
	X	0.37	0.37	0.38	0.45
	$\log T_c$	7.47	7.46	7.44	7.39
	$\log \rho_c$	1.63	1.62	1.62	1.61
ζ Her A	μ	1.36	1.34	1.32	1.24
	X	0.16	0.15	0.14	0.14
	$\log T_c$	7.21	7.19	7.19	7.17
	$\log \rho_c$	1.11	1.11	1.11	1.11
Capella	μ	0.83	0.80	0.77	
	X	0.47	0.48	0.50	
	$\log T_c$	6.71	6.71	6.68	
	$\log \rho_c$	-0.96	-0.96	-0.96	
β Aur A	μ	1.06	1.04	0.98	0.91
	X	0.30	0.29	0.31	0.33
	$\log T_c$	7.34	7.33	7.32	7.29
	$\log \rho_c$	1.10	1.10	1.10	1.10

luminosity-radius relation can be satisfied for certain values of the hydrogen and helium content. For some of the more massive stars these models are possibly applicable. These would be the supergiant stars that are quite centrally condensed. The Trumpler stars appear to be outside the range covered by these models.

I wish to thank Dr. S. Chandrasekhar for many valuable discussions.

TABLE 3
 $y_c=0.01$

ξ	$u^* = \frac{\xi}{\xi_c} - \xi^*$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta^*}{du^*}$	$\frac{d\varphi^*}{du^*}$
8.956.....	0.00	$-\infty$	$-\infty$	+0.5330			
8.553.....	.02	6.3041	1.9884	+ .5330			
8.168.....	.04	5.2856	1.6753	+ .5330			
7.800.....	.06	4.6764	1.4880	+ .5330	-1.5393	7.791	8.273
7.449.....	.08	4.2345	1.3521	+ .5329	1.3950	5.973	6.342
7.114.....	.10	3.8842	1.2443	+ .5327	1.2806	4.883	5.184
6.794.....	.12	3.5919	1.1544	+ .5323	1.1851	4.156	4.412
6.488.....	.14	3.3396	1.0767	+ .5317	1.1027	3.637	3.860
6.196.....	.16	3.1166	1.0080	+ .5308	1.0298	3.249	3.445
5.917.....	.18	2.9161	0.9462	+ .5295	0.9643	2.945	3.121
5.651.....	.20	2.7334	0.8898	+ .5278	0.9046	2.701	2.860
5.396.....	.22	2.5653	0.8378	+ .5255	0.8496	2.500	2.644
5.154.....	.24	2.4093	0.7897	+ .5227	0.7986	2.274	2.463
4.700.....	.28	2.1192	0.7080	+ .5147	0.7057	1.917	2.199
4.287.....	.32	1.8564	0.6358	+ .5031	0.6221	1.716	1.990
3.909.....	.36	1.6180	0.5703	+ .4869	0.5462	1.566	1.809
3.565.....	.40	1.4015	0.5103	+ .4653	0.4772	1.438	1.645
3.252.....	.44	1.2057	0.4552	+ .4374	0.4145	1.320	1.491
2.966.....	.48	1.0295	0.4046	+ .4025	0.3578	1.210	1.343
2.705.....	.52	0.8723	0.3583	+ .3600	0.3070	1.106	1.201
2.467.....	.56	0.7334	0.3161	+ .3097	0.2617	1.006	1.064
2.250.....	.60	0.6120	0.2778	+ .2513	0.2218	0.911	0.934
2.052.....	.64	0.5071	0.2431	+ .1850	0.1869	0.822	0.811
1.871.....	.68	0.4177	0.2119	+ .1110	0.1568	0.739	0.698
1.706.....	.72	0.3424	0.1839	+ .0300	0.1309	0.663	0.596
1.556.....	.76	0.2797	0.1588	- .0574	0.1089	0.593	0.504
1.419.....	.80	0.2282	0.1364	- .1505	0.0904	0.531	0.424
1.294.....	.84	0.1864	0.1162	- .2484	0.0749	0.476	0.354
1.181.....	.88	0.1530	0.0982	- .3504	-0.0620	0.428	0.294
1.083.....	0.9175	-0.1280	-0.0829	-0.4488			

ξ	$\eta = \frac{\xi}{\alpha_1}$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta}{d\xi}$	$\frac{d\varphi^*}{du^*}$
1.049.....	2.2	-0.1202	-0.0775	-0.4859		-0.288	
0.953.....	2.0	.0992	.0639	0.5979		.269	
0.858.....	1.8	.0803	.0517	0.7241		.248	
0.763.....	1.6	.0633	.0408	0.8677		.225	
0.667.....	1.4	.0485	.0312	1.0329		.201	
0.572.....	1.2	.0356	.0229	1.2261		.175	
0.477.....	1.0	.0247	.0159	1.4571		.148	
0.381.....	0.8	.0158	.0101	1.7426		.120	
0.286.....	0.6	.0089	.0057	2.1132		.091	
0.191.....	0.4	.0039	.0025	2.6386		.061	
0.095.....	0.2	- .0010	- .0006	3.5399		- .031	
0.000.....	0.0	0.0000	0.0000	$-\infty$		0.000	

TABLE 4

$$\gamma_c = 0.1$$

ξ	$u^* = \xi_R - \xi^*$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta^*}{du^*}$	$\frac{d\varphi^*}{du^*}$
10.183.....	0.00	$-\infty$	$-\infty$	+0.5980			
9.725.....	.02	6.7973	2.1120	+ .5980			
9.287.....	.04	5.6934	1.7752	+ .5980			
8.869.....	.06	5.0410	1.5760	+ .5980	-1.6339	8.258	8.724
8.470.....	.08	4.5709	1.4323	+ .5979	1.4821	6.301	6.659
8.089.....	.10	4.1999	1.3188	+ .5978	1.3621	5.133	5.426
7.725.....	.12	3.8912	1.2243	+ .5976	1.2623	4.358	4.608
7.377.....	.14	3.6255	1.1430	+ .5972	1.1763	3.807	4.025
7.045.....	.16	3.3910	1.0711	+ .5966	1.1003	3.395	3.588
6.728.....	.18	3.1805	1.0066	+ .5957	1.0321	3.074	3.248
6.425.....	.20	2.9889	0.9478	+ .5945	0.9700	2.817	2.975
6.136.....	.22	2.8126	0.8936	+ .5930	0.9128	2.606	2.750
5.860.....	.24	2.6492	0.8433	+ .5909	0.8597	2.430	2.562
5.344.....	.28	2.3503	0.7538	+ .5851	0.7635	2.021	2.274
4.874.....	.32	2.0734	0.6782	+ .5764	0.6766	1.804	2.078
4.445.....	.36	1.8193	0.6093	+ .5641	0.5970	1.653	1.909
4.054.....	.40	1.5865	0.5457	+ .5471	0.5238	1.528	1.753
3.697.....	.44	1.3738	0.4870	+ .5246	0.4567	1.412	1.605
3.372.....	.48	1.1807	0.4327	+ .4957	0.3954	1.302	1.460
3.075.....	.52	1.0067	0.3827	+ .4598	0.3398	1.197	1.319
2.805.....	.56	0.8514	0.3369	+ .4161	0.2898	1.094	1.180
2.558.....	.60	0.7145	0.2951	+ .3644	0.2453	0.995	1.045
2.333.....	.64	0.5953	0.2572	+ .3045	0.2061	0.900	0.916
2.128.....	.68	0.4929	0.2231	+ .2365	0.1720	0.810	0.794
1.940.....	.72	0.4061	0.1923	+ .1610	0.1425	0.726	0.681
1.770.....	.76	0.3336	0.1649	+ .0784	0.1174	0.649	0.579
1.614.....	.80	0.2740	0.1403	- .0104	0.0961	0.580	0.488
1.472.....	.84	0.2256	0.1184	- .1046	-0.0782	0.517	0.408
1.343.....	0.8797	-0.1871	-0.0990	-0.2023			

ξ	$\eta = \frac{\xi}{\alpha_1}$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta}{d\xi}$	$\frac{d\varphi^*}{du^*}$
1.334.....	1.5	-0.1851	-0.0978	-0.2100		-0.273	
1.245.....	1.4	.1614	.0851	0.2862		.262	
1.156.....	1.3	.1393	.0732	0.3700		.249	
1.067.....	1.2	.1187	.0623	0.4624		.235	
0.978.....	1.1	.0999	.0523	0.5647		.220	
0.890.....	1.0	.0826	.0432	0.6788		.203	
0.801.....	0.9	.0670	.0350	0.8068		.186	
0.712.....	0.8	.0529	.0276	0.9520		.168	
0.623.....	0.7	.0406	.0211	1.1186		.149	
0.534.....	0.6	.0298	.0155	1.3131		.129	
0.445.....	0.5	.0207	.0108	1.5452		.109	
0.356.....	0.4	.0133	.0069	1.8315		.088	
0.267.....	0.3	.0075	.0039	2.2028		.066	
0.178.....	0.2	.0033	.0017	2.7286		.044	
0.089.....	0.1	-.0008	-.0004	3.6302		-.022	
0.000.....	0.0	0.0000	0.0000	$-\infty$		0.000	

TABLE 5

$$\gamma_c = 0.25$$

ξ	$\frac{u^*}{\xi_R^* - \xi^*}$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta^*}{du^*}$	$\frac{d\varphi^*}{du^*}$
14.14.....	0.00	$-\infty$	$-\infty$	+0.6940			
13.50.....	.02	8.4218	2.5451	.6940			
12.90.....	.04	7.0292	2.1287	.6940			
12.32.....	.06	6.2362	1.8906	.6940			
11.76.....	.08	5.6758	1.7219	.6940	-1.7851	7.330	7.654
11.23.....	.10	5.2403	1.5907	.6940	1.6479	5.908	6.176
10.73.....	.12	4.8818	1.4824	.6939	1.5347	4.975	5.205
10.25.....	.14	4.5754	1.3898	.6938	1.4378	4.318	4.520
9.783.....	.16	4.3069	1.3086	.6937	1.3528	3.829	4.011
9.343.....	.18	4.0668	1.2359	.6934	1.2766	3.453	3.618
8.923.....	.20	3.8492	1.1700	.6931	1.2075	3.154	3.306
8.521.....	.22	3.6495	1.1094	.6926	1.1440	2.911	3.052
8.137.....	.24	3.4646	1.0533	.6920	1.0851	2.709	2.840
7.771.....	.26	3.2921	1.0009	.6911	1.0302	2.539	2.661
7.421.....	.28	3.1302	0.9516	.6900	0.9785	2.393	2.508
7.087.....	.30	2.9774	0.9051	.6886	0.9297	2.266	2.374
6.926.....	.31	2.9037	0.8828	.6878	0.9063	2.162	2.316
6.317.....	.35	2.6100	0.8033	.6836	0.8171	1.936	2.175
5.761.....	.39	2.3310	0.7293	.6774	0.7327	1.800	2.053
5.254.....	.43	2.0684	0.6596	.6684	0.6529	1.692	1.937
4.792.....	.47	1.8226	0.5940	.6558	0.5778	1.592	1.819
4.370.....	.51	1.5939	0.5322	.6388	0.5074	1.495	1.699
3.986.....	.55	1.3826	0.4744	.6164	0.4420	1.398	1.574
3.635.....	.59	1.1892	0.4204	.5877	0.3816	1.299	1.446
3.315.....	.63	1.0140	0.3704	.5520	0.3263	1.200	1.314
3.023.....	.67	0.8574	0.3244	.5084	0.2764	1.100	1.181
2.757.....	.71	0.7193	0.2824	.4566	0.2319	1.001	1.048
2.515.....	.75	0.5993	0.2443	.3965	0.1926	0.905	0.918
2.293.....	.79	0.4967	0.2100	.3283	0.1583	0.813	0.795
2.092.....	.83	0.4105	0.1792	.2523	0.1289	0.726	0.680
1.908.....	.87	0.3393	0.1518	.1694	0.1038	0.646	0.576
1.740.....	.91	0.2815	0.1275	.0806	-0.0826	0.574	0.484
1.689.....	0.9228	-0.2656	-0.1203	+0.0510			

ξ	$\eta = \frac{\xi}{a_1}$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta}{d\xi}$	$\frac{d\varphi^*}{du^*}$
1.546.....	1.0	-0.2236	-0.1009	-0.0406		-0.237	
1.391.....	0.9	.1819	.0818	0.1539		.223	
1.237.....	0.8	.1442	.0647	0.2855		.207	
1.082.....	0.7	.1108	.0496	0.4400		.188	
0.928.....	0.6	.0817	.0365	0.6237		.166	
0.773.....	0.5	.0569	.0254	0.8466		.142	
0.618.....	0.4	.0365	.0163	1.1252		.116	
0.464.....	0.3	.0206	.0092	1.4905		.089	
0.309.....	0.2	.0092	.0041	2.0119		.060	
0.155.....	0.1	-.0023	-.0010	2.9109		-.030	
0.000.....	0.0	0.0000	0.0000	$-\infty$		0.000	

TABLE 6
 $y_c=0.5$

ξ	$u^* = \xi_R - \xi^*$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta^*}{du^*}$	$\frac{d\varphi^*}{du^*}$
43.23.....	0.00	$-\infty$	$-\infty$	+0.8340			
39.42.....	0.04	11.9641	3.4968	.8340			
35.96.....	0.08	9.9364	2.9103	.8340			
32.79.....	0.12	8.7564	2.5671	.8340			
29.91.....	0.16	7.9155	2.3216	.8340			
27.28.....	0.20	7.2564	2.1305	.8340	-2.1779	4.363	4.458
24.88.....	0.24	6.7102	1.9680	.8340	2.0137	3.703	3.791
22.69.....	0.28	6.2404	1.8296	.8340	1.8720	3.239	3.322
20.69.....	0.32	5.8257	1.7072	.8339	1.7463	2.897	2.976
18.87.....	0.36	5.4524	1.5969	.8338	1.6328	2.634	2.710
17.21.....	0.40	5.1114	1.4958	.8336	1.5288	2.426	2.499
15.70.....	0.44	4.7962	1.4023	.8333	1.4324	2.258	2.328
14.31.....	0.48	4.5021	1.3148	.8329	1.3422	2.119	2.187
13.05.....	0.52	4.2256	1.2325	.8322	1.2572	2.002	2.068
11.91.....	0.56	3.9641	1.1545	.8313	1.1766	1.902	1.966
10.86.....	0.60	3.7157	1.0802	.8300	1.0998	1.815	1.877
9.903.....	0.64	3.4786	1.0091	.8283	1.0263	1.739	1.799
9.032.....	0.68	3.2373	0.9424	.8261	0.9552	1.620	1.776
8.237.....	0.72	2.9830	0.8777	.8232	0.8836	1.617	1.798
7.512.....	0.76	2.7279	0.8130	.8190	0.8115	1.617	1.808
6.851.....	0.80	2.4755	0.7485	.8134	0.7393	1.605	1.799
6.249.....	0.84	2.2285	0.6848	.8056	0.6678	1.578	1.772
5.699.....	0.88	1.9892	0.6225	.7951	0.5978	1.537	1.726
5.197.....	0.92	1.7598	0.5620	.7810	0.5299	1.484	1.663
4.740.....	0.96	1.5424	0.5039	.7624	0.4649	1.420	1.584
4.323.....	1.00	1.3389	0.4485	.7384	0.4034	1.346	1.490
3.942.....	1.04	1.1510	0.3964	.7080	0.3459	1.262	1.383
3.596.....	1.08	0.9802	0.3477	.6703	0.2929	1.172	1.265
3.279.....	1.12	0.8275	0.3027	.6247	0.2448	1.076	1.139
2.991.....	1.16	0.6934	0.2616	.5706	0.2018	0.979	1.011
2.728.....	1.20	0.5781	0.2244	.5082	0.1640	0.881	0.883
2.488.....	1.24	0.4808	0.1910	.4377	0.1311	0.787	0.761
2.269.....	1.28	0.4005	0.1613	.3600	0.1030	0.699	0.648
2.180.....	1.2974	-0.3705	-0.1495	+0.3241	-0.0921	0.662	0.600

ξ	$\eta = \frac{\xi}{a_1}$	$\sigma^*(\xi)$	$\theta^*(\xi)$	$\psi^*(\xi)$	$\varphi^*(\xi)$	$\frac{d\theta}{d\xi}$	$\frac{d\varphi^*}{du^*}$
2.028.....	0.60	-0.3225	-0.1304	+0.2563		-0.213	
1.859.....	.55	.2728	.1101	+0.1711		.206	
1.690.....	.50	.2269	.0914	+0.0731		.197	
1.521.....	.45	.1849	.0744	-0.0399		.186	
1.352.....	.40	.1469	.0590	-0.1713		.173	
1.183.....	.35	.1131	.0454	-0.3254		.157	
1.014.....	.30	.0834	.0335	-0.5088		.139	
0.845.....	.25	.0582	.0233	-0.7314		.120	
0.676.....	.20	.0374	.0150	-1.0097		.098	
0.507.....	.15	.0211	.0084	-1.3748		.075	
0.338.....	.10	.0094	.0038	-1.8961		.051	
0.169.....	.05	-.0023	-.0009	-2.7950		-.026	
0.000.....	0.00	0.0000	0.0000	$-\infty$		0.000	

